WHAT HAS ALGEBRAIC GEOMETRY TO DO WITH GEOMETRY?

BENJAMIN SCHMIDT

Abstract. Algebraic geometry has found its way into many areas of pure mathematics such as number theory, representation theory and others. This has been possible in particular due to the extremely general foundations laid out by Grothendieck. From this point of view the geometric roots of the field are clouded and only barely discussed in many first lectures. We will exclusively talk about elementary questions in complex algebraic geometry that have their origins in 19th century mathematics. These are notes for a talk given in the graduate seminar of the Ohio State University in April 2016.

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1. What is Algebraic Geometry

Algebraic geometry is often described as the study of solutions to systems of polynomial equations. This is technically true, but it gives the wrong impression. Rarely do algebraic geometers explicitly describe the solution set to some equations. In these notes we will take the point of view that we want to study certain complex manifolds. Miraculously it will turn out that they can be described by polynomial equations.

Set theoretically *complex projective* space $\mathbb{P}^n$ for $n \in \mathbb{N}$ is given by all lines (complex one dimensional subspaces) $L \subset \mathbb{C}^{n+1}$. An element $x \in \mathbb{P}^n$ will be described by its *homogeneous coordinates* $x = (x_0 : \ldots : x_n)$, where $(x_0, \ldots, x_n)$ is a generator of L and for any $\lambda \in \mathbb{C}^*$ one has $(x_0 : \ldots : x_n) = (\lambda x_0 : \ldots : \lambda x_n)$. Projective space has a topology inherited from the euclidean topology on $\mathbb{C}^{n+1}$. We want to give $\mathbb{P}^n$ the structure of a complex manifold, i.e. we need to cover it with open subsets of $\mathbb{C}^n$. For $i = 0, \ldots n$, we define a covering with open subsets by

$$\begin{aligned} U_i &= \{(x_0 : \ldots : x_n) \in \mathbb{P}^n : x_i \neq 0\} \\ &= \left\{ \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right) \in \mathbb{C}^{n+1} \right\} = \mathbb{C}^n. \end{aligned}$$

One can show that $\mathbb{P}^n$ is compact. The goal of complex algebraic geometry is to study closed complex submanifolds of $\mathbb{P}^n$.

Definition 1.1. A *complex projective manifold* is a connected compact complex manifold X that has a closed embedding $X \hookrightarrow \mathbb{P}^n$ for some $n \in \mathbb{N}$.

The following theorem says that any complex projective manifold is cut out by polynomial equations. Unfortunately, it was not proven in the 19th century. Recall that a homogeneous polynomial is a polynomial in which all monomials have the same degree. Since coordinates in projective space are only well defined up to scale, only homogeneous polynomials have a well defined vanishing locus.

Theorem 1.2 (Chow's Theorem [Cho57]). *Let $X \hookrightarrow \mathbb{P}^n$ be an embedded complex projective manifold. Then there are finitely many homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ such that*

$$X = \{x \in \mathbb{P}^n : f_1(x) = \ldots = f_r(x) = 0\}.$$

2. Lines on Hypersurfaces

Wake an algebraic geometer in the dead of night, whispering: 27. Chances are, he will respond: "lines on a cubic surface". [DS81]

A line in $\mathbb{P}^n$ is an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ such that its image is cut out by linear equations. This is potentially the easiest example of a projective manifold one can think of. We will encounter non linear embeddings of $\mathbb{P}^1$ in the next section. Another important class of examples are so called *hypersurfaces*. Those are submanifolds of $\mathbb{P}^n$ cut out by a single polynomial equation. In 1849 Cayley and Salmon proved one of the most famous results in algebraic geometry.

Theorem 2.1 (Cayley-Salmon Theorem 1849). *Let $X_3 \subset \mathbb{P}^3$ be a projective manifold cut out by a single degree three equation. Then there are exactly 27 lines in $\mathbb{P}^3$ that are contained in X_3.*

For those who know what this means: The 27 lines naturally form a basis of the fundamental representation of the Weyl group of E_6.

We will give a proof in the special case of the cubic $X \subset \mathbb{P}^3$ cut out by $f(x, y, z, w) = x^3 + y^3 + z^3 + w^3$. In that case all the lines are given by equations of the form $x - \lambda w = 0 = y - \mu z$, where $\lambda^3 = \mu^3 = -1$ and permutations of the coordinates (count that these are 27!).

Proof. The equation f is invariant under permuting the coordinates x, y, z, w. Therefore, if a line is in X, then permuting the coordinates will give another line in X. Using Gaussian elimination and permuting the coordinates, we can assume that there are $a, b, c, d \in \mathbb{C}$ such that our line is given by $x = az + bw$ and $y = cz + dw$. In order to be contained in the cubic surface X, we need $f(az + bw, cz + dw, z, w) = 0$ as a polynomial in $\mathbb{C}[z, w]$. That means its coefficients have to vanish which can be computed as

$$\begin{aligned} a^3 + c^3 &= -1 \\ b^3 + d^3 &= -1 \\ a^2 b + c^2 d &= 0 \\ ab^2 + cd^2 &= 0. \end{aligned}$$

If $a = 0$ holds, then $d = 0$, $b^3 = -1$ and $c^3 = -1$ as claimed. Assume that $a \neq 0$ holds. Then we get

$$(a^2 b)^2 = (c^2 d)^2 = c^4 d^2 = -ab^2 c^3 = ab^2(a^3 + 1) = a^4 b^2 + ab^2.$$

This implies $b = 0$ and the same argument as for $a = 0$ finishes the proof. □

Notice that we have four equations with four variables controlling the number of curves in the previous proof. This heuristically suggests that the number should be finite. Assume that X is a hypersurface in $\mathbb{P}^4$ of degree 5. Then we will get six equations with six variables in the same way and would expect the number to be finite. Indeed, it is classically known that most quintic threefolds have 2875 lines, but in special cases there can be infinitely many.

More generally, linear subspaces in projective space for fixed dimensions form a variety known as the Grassmannian. A whole field known as Schubert calculus has formed around the enumerative questions of linear subspaces.

3. Twisted Cubics

There is a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by $(a : b) \mapsto (a^3 : a^2b : ab^2 : b^3)$. We can define a curve $C \cong \mathbb{P}^1$ as the image of this map. This is known as a twisted cubic curve. Note that the map depends on the choice of coordinates in $\mathbb{P}^3$, which in turns corresponds to a basis of $\mathbb{C}^{n+1}$ up to scaling. Therefore, we can encode such a choice of coordinates in the group $\mathrm{PGL}(4) = GL(4)/C^*$. By definition PGL(4) acts transitively on the set of twisted cubics. Computing the stabilizer for a single twisted cubic such as the one above will lead to a bijection

$$\{\text{twisted cubics}\} \cong \mathrm{PGL}(4)/\mathrm{PGL}(2).$$

We call this the *moduli space* of twisted cubic curves. Since $\dim \mathrm{PGL}(n) = n^2 - 1$, this space has dimension $15 - 3 = 12$. Take a line L in $\mathbb{P}^3$. What is the codimension of the locus of twisted cubics that intersect L non trivially? First of all twisted cubics are not contained in any plane. This can be computed explicitly with the twisted cubic above. Choose an arbitrary plane H containing L. Then L intersects C if and only if one of the finitely many points in $H \cap C$ is in L. A point in a plane lying on a line is a codimension 1 condition.

If we choose 12 lines, the locus of twisted cubics intersecting all 12 lines is the intersection of 12 codimension 1 loci. Therefore, we would expect this number to be finite. Obviously, if all lines are the same this is not true. More generally, the conditions for this to be true are more subtle, but it turns out to work for most configurations of lines. Schubert wrote down the number in 1879 in his book [Sch79]. However, he did not give a proof and it took until 1995 to be fully resolved.

Theorem 3.1 ([ES95, Sch79])**.** *Given 12 general lines in $\mathbb{P}^3$, there are 80160 twisted cubics intersecting them.*

Schubert gave many such numbers without proof. In fact, Hilbert's 15th problem was to give a "Rigorous foundation of Schubert's enumerative calculus".

4. Higher Degree Rational Curves

The questions about lines on hypersurfaces can be generalized by asking about so called *rational curves*, i.e. curves that are isomorphic to $\mathbb{P}^1$. For example twisted cubics are such rational curves that are not lines. In order to make sense of these questions we need to define what the degree of a curve in projective space is. The following theorem is a special case of what is more generally Bézout's Theorem.

Theorem 4.1. *Let $C \subset \mathbb{P}^n$ be a curve and $H \subset \mathbb{P}^n$ be a hyperplane. Then either $C \subset H$ or $C \cap H$ is finite. Moreover, the number of intersection points in $C \cap H$ is constant (counted with appropriate multiplicity) for all hyperplanes H such that C is not contained in H.*

This theorem shows that the following notion is well defined

Definition 4.2. Let $C \subset \mathbb{P}^n$ be a curve. Then the number of points d of the intersection of C with a general hyperplane is called the *degree* of C.

Exercise 4.3. Show that lines have degree one and twisted cubics have degree three.

Let $X_5 \subset \mathbb{P}^4$ be a quintic hypersurface. How many rational curves of a fixed degree d are there on X_5? This question has lead to a lot of work by various mathematicians. The first obvious question is whether this number is even finite. This is wrong in general, but conjectured by Herb Clemens in [Cle84] to be true for a general enough quintic.

Conjecture 4.4 (Clemens Conjecture)**.** *The number n_d of degree d rational curves on a very general degree five hypersurface in $\mathbb{P}^4$ is finite.*

The conjecture is based on a dimension count heuristic as before. Sheldon Katz proved the conjecture for $d \leq 7$ in [Kat86]. As of now it is known for $d \leq 11$ with the most recent case in [Cot12].

Not knowing this conjecture has not stopped people to compute these numbers n_d. The question received an extraordinary interest due to its surprising connection with theoretical physics. In [COGP91] string theorists gave a heuristic way to compute these numbers using intuition coming from physics. Mathematicians were both sceptical and intrigued. In the example of twisted cubics they claimed that the number of twisted cubics lying on a very general quintic threefold is given by $317,206,375$. In the early 90's Ellingsrud and Strømme gave a mathematical proof to compute this number. At the end of their argument they used a computer to compute the actual numbers. Their computation came out as $2,682,549,425$ disagreeing with the physicists. Many algebraic geometers who were already sceptical assumed the physicists must have made a mistake. However, it turned out the computer calculations contained a programming error. Finally, Ellingsrud and Strømme arrived at the same number in [ES95].

Confirming the computations done by the string theorists became known as the mirror conjecture. Givental showed that the numbers are correct provided the Clemens Conjecture holds in [Giv96].

5. On the Genus of a Curve

So far we talked about the degree of a curve. Another important invariant that is independent of the embedding into $\mathbb{P}^n$ is the so called *genus*. It is colloquially defined as the number of "holes"

Figure 1. Surfaces of genus one, two and three by Oleg Alexandro using MATLAB.

in the real surface, obtained by forgetting the complex structure. The following fact is contained in most introductory texts on algebraic topology.

Theorem 5.1. *Let S be a real orientable manifold of dimension two. Then the Euler characteristic of S is given by $\chi(S) = 2 - 2g(S)$ for some non negative integer $g(S)$.*

Recall (or learn if you have never heard it) that any complex manifold is orientable as a real manifold. For any curve C we define its *genus* to be $g(C)$. A classical question in algebraic geometry is the following.

Question 5.2. For which non-negative pairs of integers (d, g) is there a curve in $\mathbb{P}^3$ of degree d and genus g?

An answer to this question was given by Halphen in 1882. Later it turned out that his proof contained a mistake, but the result was true nonetheless as shown by Gruson and Peskine in 1982. We took the statement of the theorem from [Har87].

Theorem 5.3 ([Hal82, GP82]). *Let C be a curve in $\mathbb{P}^3$ of degree d and genus g. Then the following holds.*

(1) *If C is contained in a plane, then $g = \frac{1}{2}(d-1)(d-2)$.*
(2) *If C is contained in a quadric surface, then there are positive integers a, b such that $d = a+b$ and $g = (a-1)(b-1)$. For any such a, b there is an actual curve with these invariants.*
(3) *If C is not contained in a quadric or plane, then the inequality*

$$0 \leq g \leq \frac{1}{6}d(d-3) + 1$$

holds. Moreover, for any (d, g) satisfying these points there is a curve with these invariants.

Part (1) and (2) of the Theorem suggest a different question.

Question 5.4. Let C be a curve in $\mathbb{P}^3$ of degree d. Assume further that C is not contained in any surface of degree smaller than k. What is the maximal genus that such a curve can attain?

While there has been a lot of progress on this question in special cases, it is still open today in full generality. We can recommend [Har87] for a further overview on this question.

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Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210-1174, USA

E-mail address: schmidt.707@osu.edu

URL: https://people.math.osu.edu/schmidt.707/