# A GENERALIZED BOGOMOLOV-GIESEKER INEQUALITY FOR THE SMOOTH QUADRIC THREEFOLD

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Abstract. We prove a generalized Bogomolov-Gieseker inequality as conjectured by Bayer, Macrì and Toda for the smooth quadric threefold. This implies the existence of a family of Bridgeland stability conditions.

## 1. Introduction

The classical notion of slope stability has been explored for a long time to study vector bundles and their moduli spaces. One important direction of study is the birational geometry of a given moduli space. Historically, an approach for obtaining divisorial contractions or flips was varying the polarization of the variety and therefore varying the GIT problem. However, this does not provide enough flexibility. For example, if the Picard group is $\mathbb{Z}$, there is no possible variation.

Inspired by the study of Dirichlet branes in string theory by Douglas (see [Dou00, Dou01, Dou02]), the notion of Bridgeland stability was introduced in [Bri07]. Instead of defining stability in the category of coherent sheaves, one uses other abelian categories inside the bounded derived category of coherent sheaves. Bridgeland shows that the set of all these stability conditions forms a complex manifold. This leads to plenty of room to vary a given stability condition even if the Picard rank is 1.

While this notion provides many of the desired properties, constructing such Bridgeland stability conditions has turned out to be a serious issue. A large family was constructed in the case of K3 surfaces in [Bri08]. Arcara and Bertram generalized this construction to any smooth complex projective surface in [AB13]. Examples of successful applications are found in the birational geometry of Hilbert schemes of points on smooth projective surfaces (see for example [ABCH13, BM13, MM13, YY14]). Toda shows that the minimal model program on any smooth projective surface is realized as a variation of moduli spaces of Bridgeland stable objects in [Tod12].

The case of threefolds seems to be more complicated. The work of Bridgeland was motivated by the case of Calabi-Yau threefolds occurring in string theory. So far no Bridgeland stability condition has been constructed on a single Calabi-Yau threefold. A promising approach for all smooth projective

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threefolds is due to Bayer, Macrì and Toda in [BMT14]. It was confirmed to work for $\mathbb{P}^3$ in [Mac12] and for principally polarized abelian threefolds of Picard rank one in [MP13a, MP13b]. By mimicking the construction for surfaces, Bayer, Macrì and Toda obtain the notion of tilt-stability on an abelian category $\mathcal{B}^{\omega,B}$ in the bounded derived category of coherent sheaves for any $\mathbb{R}$-divisor $B$ and any ample $\mathbb{R}$-divisor $\omega$. The slope function is given by

$$\nu_{\omega,B} := \frac{\omega \operatorname{ch}_2^B - \frac{\omega^3}{2} \operatorname{ch}_0^B}{\omega^2 \operatorname{ch}_1^B},$$

where $\operatorname{ch}^B = e^{-B} \operatorname{ch}$. Unlike in the case of surfaces this provides no Bridgeland stability condition. They conjecture a generalized Bogomolov-Gieseker inequality on third Chern classes for tilt-stable objects $E \in \mathcal{B}^{\omega,B}$ which satisfy $\nu_{\omega,B}(E) = 0$ given by

$$\operatorname{ch}_3^B(E) \leq \frac{\omega^2}{6} \operatorname{ch}_1^B(E).$$

This inequality turns out to be the missing ingredient for the construction of Bridgeland stability conditions. Interestingly, there are other applications of this inequality besides the construction of Bridgeland stability conditions. One of the most interesting consequences is Fujita's conjecture (see [BBMT11]). Macrì was able to prove the inequality in the case of $\mathbb{P}^3$ in [Mac12], while Maciocia and Piyaratne managed to show it for principally polarized abelian threefolds of Picard rank one in [MP13a, MP13b]. The main result of this article is the following.

**Theorem 1.1.** *(See Theorem 4.1) The generalized Bogomolov-Gieseker inequality is true for the smooth quadric threefold $Q$. In particular, there is a large family of Bridgeland stability conditions on $Q$.*

The proof is based on calculations with a strong full exceptional collection in $D^b(Q)$ that exists due to [Kap88]. We break it down to a technical lemma from [BMT14] (see Proposition 4.2).

The paper is organized as follows. Basics on stability and the construction of [BMT14] are explained in Section 2. In Section 3 some facts about the smooth quadric threefold are being recalled. Finally, Section 4 deals with the proof of the main theorem.

**Notation.** By $X$ we denote a smooth projective threefold over the complex numbers. Its bounded derived category of coherent sheaves is called $D^b(X)$. Let $Q$ be the smooth quadric threefold in $\mathbb{P}^4$ over the complex numbers defined by the equation $x_0^2 + x_1x_2 + x_3x_4 = 0$.

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## 2. Construction of Stability Conditions

Let us recall some definitions concerning stability. The central part of the theory is the notion of Bridgeland stability conditions that was introduced in [Bri07]. Let $H := \{re^{i\pi\varphi} : r > 0, \varphi \in (0,1]\}$ be the upper half plane plus the negative real line. A Bridgeland stability condition on $D^b(X)$ is a pair $(Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure and $Z : K_0(X) = K_0(\mathcal{A}) \to \mathbb{C}$ is a homomorphism such that $Z(\mathcal{A}\backslash\{0\}) \subset H$ holds plus a technical property.

The inclusion $Z(\mathcal{A}\backslash\{0\}) \subset H$ turns out to be the crucial point for threefolds. Note that for any smooth projective variety of dimension bigger than or equal to two, there is no Bridgeland stability condition factoring through the Chern character for $\mathcal{A} = \mathrm{Coh}(X)$ due to [Tod09, Lemma 2.7].

In order to construct such stability conditions on a smooth projective threefold $X$, Bayer, Macrì and Toda proposed a construction in [BMT14]. We will review it. Let $B$ be any $\mathbb{R}$-divisor. Then the twisted Chern character $\mathrm{ch}^B$ is defined to be $e^{-B}\,\mathrm{ch}$. In more detail, we have

$$\begin{aligned}
\mathrm{ch}_0^B &= \mathrm{ch}_0,\\
\mathrm{ch}_1^B &= \mathrm{ch}_1 - B\,\mathrm{ch}_0,\\
\mathrm{ch}_2^B &= \mathrm{ch}_2 - B\,\mathrm{ch}_1 + \frac{B^2}{2}\,\mathrm{ch}_0,\\
\mathrm{ch}_3^B &= \mathrm{ch}_3 - B\,\mathrm{ch}_2 + \frac{B^2}{2}\,\mathrm{ch}_1 - \frac{B^3}{6}\,\mathrm{ch}_0 .
\end{aligned}$$

The category of coherent sheaves $\mathrm{Coh}(X)$ is the heart of a bounded t-structure on $D^b(X)$. Let $\omega$ be an ample $\mathbb{R}$-divisor. Then we can define a twisted version of the standard slope stability function on $\mathrm{Coh}(X)$ by

$$\mu_{\omega,B} := \frac{\omega^2\,\mathrm{ch}_1^B}{\omega^3\,\mathrm{ch}_0^B},$$

where dividing by 0 is interpreted as $+\infty$. The process of tilting is used to obtain a new heart of a bounded t-structure. For more information on this general theory we refer to [HRS96]. A torsion pair is defined by

$$\begin{aligned}
\mathcal{T}_{\omega,B} &= \{E \in \mathrm{Coh}(X) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_{\omega,B}(G) > 0\},\\
\mathcal{F}_{\omega,B} &= \{E \in \mathrm{Coh}(X) : \text{any subsheaf } F \subset E \text{ satisfies } \mu_{\omega,B}(F) \leq 0\}.
\end{aligned}$$

A new heart of a bounded t-structure is defined by the extension closure $\mathcal{B}^{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B}\rangle$. A new slope function is defined by

$$\nu_{\omega,B} := \frac{\omega\,\mathrm{ch}_2^B - \frac{\omega^3}{2}\,\mathrm{ch}_0^B}{\omega^2\,\mathrm{ch}_1^B},$$

where dividing by 0 is again interpreted as $+\infty$. Note that in regard to [BMT14] this slope has been modified by switching $\omega$ with $\sqrt{3}\omega$. We prefer this point of view because it will slightly simplify a few computations. On smooth projective surfaces the map $-\operatorname{ch}_2^B + \frac{\omega^2}{2}\operatorname{ch}_0^B + i\omega\operatorname{ch}_1^B$ from $\mathcal{B}^{\omega,B}(X)$ to $\mathbb{C}$ is already a Bridgeland stability function (see [Bri08, AB13]). However, on threefolds this is not enough. For example skyscraper sheaves are still mapped to the origin. Therefore, Bayer, Macrì and Toda propose another analogous tilt via

$$\mathcal{T}'_{\omega,B} = \{E \in \mathcal{B}^{\omega,B}(X) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \nu_{\omega,B}(G) > 0\},$$
$$\mathcal{F}'_{\omega,B} = \{E \in \mathcal{B}^{\omega,B}(X) : \text{any subobject } F \hookrightarrow E \text{ satisfies } \nu_{\omega,B}(F) \leq 0\}$$

and setting $\mathcal{A}^{\omega,B} := \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B}\rangle$. Finally, they define for any $s > 0$ functions by

$$Z_{\omega,B,s} := (-\operatorname{ch}_3^B + s\omega^2\operatorname{ch}_1^B) + i(\omega\operatorname{ch}_2^B - \frac{\omega^3}{2}\operatorname{ch}_0^B),$$
$$\lambda_{\omega,B,s} := -\frac{\Re(Z_{\omega,B,s})}{\Im(Z_{\omega,B,s})}.$$

The function $\lambda_{\omega,B,s}$ is called the slope of $Z_{\omega,B,s}$.

**Definition 2.1.** An object $E \in \mathcal{B}^{\omega,B}$ is called $\nu_{\omega,B}$*-(semi)stable* (or *tilt-(semi)stable*) if for any exact sequence $0 \to F \to E \to G \to 0$ the inequality $\nu_{\omega,B}(F) < (\leq)\nu_{\omega,B}(G)$ holds.

The following theorem motivates the whole construction.

**Theorem 2.2** ([BMT14, Corollary 5.2.4])**.** *Let $X$ be a smooth projective threefold over the complex numbers, $\omega$ an ample divisor, $B$ any divisor and $s > 0$. Then $(Z_{\omega,B,s}, \mathcal{A}^{\omega,B})$ is a Bridgeland stability condition if and only if for any $\nu_{\omega,B}$-stable object $E \in \mathcal{B}^{\omega,B}$ with $\nu_{\omega,B}(E) = 0$ the inequality*

$$\operatorname{ch}_3^B(E) < s\omega^2\operatorname{ch}_1^B(E) \tag{1}$$

*holds.*

The inequality (1) in the theorem is exactly expressing the fact that $Z_{\omega,B,s}$ is not mapping to the non-negative real line $\mathbb{R}_{\geq 0}$. Bayer, Macrì and Toda hope that (1) holds for $s = 3/2$. They even conjecture a stronger inequality.

**Conjecture 2.3** ([BMT14, Conjecture 1.3.1])**.** *Inequality* (1) *holds for all* $s > \frac{1}{6}$.

## 3. Quadric Threefold

In order to prove Conjecture 2.3 for the smooth quadric threefold $Q$, we need to recall some facts about its bounded derived category of coherent sheaves $D^b(Q)$. In the following, we view $Q$ as being cut out by the equation $x_0^2 + x_1x_2 + x_3x_4 = 0$ in $\mathbb{P}^4$.

Since the open subvariety of $Q$ defined by $x_1 \neq 0$ is isomorphic to $\mathbb{A}^3$, the Picard group of $Q$ is isomorphic to $\mathbb{Z}$ and is generated by a very ample line bundle $\mathcal{O}(H)$. Moreover, the equality $H^3 = 2$ holds because a general line in $\mathbb{P}^4$ intersects $Q$ in two points.

Let us recall exceptional collections.

**Definition 3.1.** A *strong exceptional collection* is a sequence $E_1, \ldots, E_r$ of objects in $D^b(X)$ such that $\operatorname{Ext}^i(E_l, E_j) = 0$ for all $l, j$ and $i \neq 0$, $\operatorname{Hom}(E_j, E_j) = \mathbb{C}$ and $\operatorname{Hom}(E_l, E_j) = 0$ for all $l > j$. Moreover, it is called *full* if $E_1, \ldots, E_r$ generates $D^b(X)$ via shifts and extensions.

On $Q$ line bundles are not enough to obtain a full strong exceptional collection. Therefore, we need to introduce the spinor bundle $S$. We refer to [Ott88] for a more detailed treatment. The spinor bundle is defined via an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \to \iota_* S \to 0$$

where $\iota : Q \hookrightarrow \mathbb{P}^4$ is the inclusion and the first map is given by a matrix $M$ such that $M^2 = (x_0^2 + x_1 x_2 + x_3 x_4) I_4$ for the identity $4 \times 4$ matrix $I_4$. Restricting the second morphism to $Q$ leads to

$$0 \to S(-1) \to \mathcal{O}_Q^{\oplus 4} \to S \to 0. \tag{2}$$

Due to Kapranov (see [Kap88])

$$\mathcal{O}(-1), S(-1), \mathcal{O}, \mathcal{O}(1)$$

is a strong full exceptional collection on $D^b(Q)$.

Explicit computations lead to a resolution of the skyscraper sheaf $k(x)$ given by

$$0 \to \mathcal{O}(-1) \to S(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{O}(1) \to k(x) \to 0 \tag{3}$$

for any $x \in Q$.

## 4. Main Result

The main result of this article is the following.

**Theorem 4.1.** *Conjecture 2.3 holds for the smooth projective threefold $Q$, i.e., for any $\nu_{\omega,B}$-stable object $E \in \mathcal{B}^{\omega,B}$ with $\nu_{\omega,B}(E) = 0$ the inequality*

$$\operatorname{ch}_3^B(E) \leq \frac{\omega^2}{6} \operatorname{ch}_1^B(E)$$

*holds.*

There are $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that $\omega = \alpha H$ and $B = \beta H$. Therefore, we will replace $B$ by $\beta$ and $\omega$ by $\alpha$ in the notation of slope functions and categories. Due to Proposition 2.7 and Lemma 3.2 in [Mac12] it suffices to prove the statement for $\alpha < \frac{1}{3}$ and $\beta \in [-\frac{1}{2}, 0]$.

The following technical proposition provides the basis of the proof.

**Proposition 4.2** ([BMT14, Lemma 8.1.1]). *Let $\mathcal{C} \subset D^b(X)$ be the heart of a bounded t-structure with the following properties.*

(i) *There exists* $\phi_0 \in (0,1)$ *and* $s_0 \in \mathbb{Q}$ *such that*

$$Z_{\alpha,\beta,s_0}(\mathcal{C}) \subset \{re^{\pi\phi i} : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1\}.$$

(ii) *The inclusion* $\mathcal{C} \subset \langle \mathcal{A}_{\alpha,\beta}, \mathcal{A}_{\alpha,\beta}[1] \rangle$ *holds.*

(iii) *For all points* $x \in X$ *we have* $k(x) \in \mathcal{C}$ *and for all proper subobjects* $C \hookrightarrow k(x)$ *in* $\mathcal{C}$ *the inequality* $\Im Z_{\alpha,\beta,s_0}(C) > 0$ *holds.*

*Then the pair* $(Z_{\alpha,\beta,s}, \mathcal{A}_{\alpha,\beta})$ *is a stability condition on* $D^b(X)$ *for all* $s > s_0$*.*

Due to [Bon90] a full strong exceptional collection induces an equivalence between $D^b(X)$ and the bounded derived category of finitely generated modules over some finite dimensional algebra $A$. In the special case of the smooth quadric $Q$, we get the heart of a bounded t-structure by setting

$$\mathcal{C} := \langle \mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1) \rangle.$$

Moreover, $\mathcal{C}$ is isomorphic to the category of finitely generated modules over some finite dimensional algebra $A$ and $\mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1)$ are the simple objects.

We will show that the conditions of the lemma are fulfilled for this $\mathcal{C}$ and $s_0 = \frac{1}{6}$. In order to do that, a computation of the values for the different slope-functions is necessary. By using (2) we can obtain the following lemma.

**Lemma 4.3.** *For all* $n \in \mathbb{N}$ *we have*

$$\operatorname{ch}^\beta(\mathcal{O}(n)) = 1 + (n-\beta)H + (n-\beta)^2 \frac{H^2}{2} + \frac{1}{3}(n-\beta)^3.$$

*The chern character of* $S(-1)$ *is given by*

$$\operatorname{ch}^\beta(S(-1)) = 2 - (2\beta+1)H + \beta(\beta+1)H^2 + \frac{1}{6} - \beta^2 - \frac{2}{3}\beta^3.$$

*We have the following* $\mu$*-slopes*

$$\mu_{\alpha,\beta}(\mathcal{O}(1)) = \frac{1-\beta}{\alpha}, \qquad \mu_{\alpha,\beta}(\mathcal{O}) = -\frac{\beta}{\alpha},$$
$$\mu_{\alpha,\beta}(\mathcal{O}(-1)) = -\frac{\beta+1}{\alpha}, \; \mu_{\alpha,\beta}(S(-1)) = -\frac{2\beta+1}{2\alpha}.$$

*The* $\nu$*-slopes for the same sheaves are given by*

$$\nu_{\alpha,\beta}(\mathcal{O}(1)) = \frac{(1-\beta)^2 - \alpha^2}{2\alpha(1-\beta)}, \quad \nu_{\alpha,\beta}(\mathcal{O}) = \frac{\alpha^2 - \beta^2}{2\alpha\beta},$$
$$\nu_{\alpha,\beta}(\mathcal{O}(-1)) = \frac{\alpha^2 - (1+\beta)^2}{2\alpha(1+\beta)}, \; \nu_{\alpha,\beta}(S(-1)) = \frac{\alpha^2 - \beta(\beta+1)}{\alpha(2\beta+1)}.$$

*Finally, the Z values can be computed as*

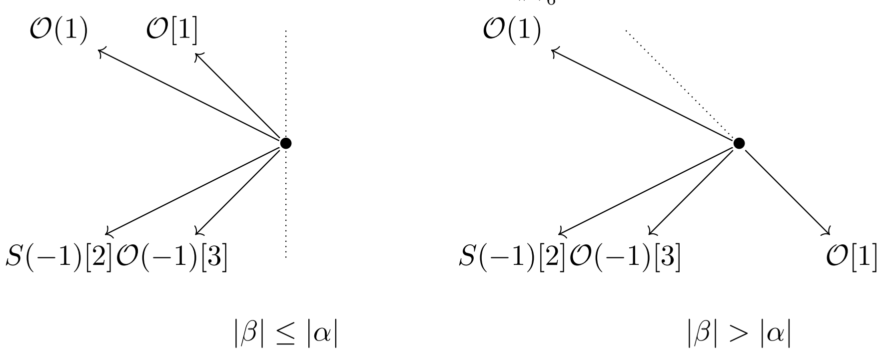
$$Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}(1)) = \frac{1}{3}((1-\beta)^2-\alpha^2)(\beta-1+3i\alpha),$$
$$Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}) = \frac{1}{3}(\beta^2-\alpha^2)(\beta+3i\alpha),$$
$$Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}(-1)) = \frac{1}{3}((1+\beta)^2-\alpha^2)(\beta+1+3i\alpha),$$
$$Z_{\alpha,\beta,\frac{1}{6}}(S(-1)) = \frac{1}{6}(2\beta+1)(2\beta^2+2\beta-1-2\alpha^2)+2i\alpha(\beta^2+\beta-\alpha^2).$$

At this point we can prove the first assumption in Proposition 4.2.

**Lemma 4.4.** *There exists* $\phi_0 \in (0,1)$ *such that*

$$Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{C}) \subset \{re^{\pi\phi i} : r \geq 0, \phi_0 \leq \phi \leq \phi_0+1\}.$$

*Proof.* It suffices to show that the 4 generators of $\mathcal{C}$ are contained in some half plane of $\mathbb{C}$. There are two different cases to deal with. Lemma 4.3 shows that the half plane of points with negative real part works if $|\beta| \leq |\alpha|$, while the half plane left of the line through 0 and $Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}[1])$ works in the case $|\beta| > |\alpha|$. The following figure shows the $Z_{\alpha,\beta,\frac{1}{6}}$ values.




□

Before we can show assumption (ii) in Proposition 4.2, we need to deal with continuity issues for tilt-stability. For any $E \in \mathcal{B}^{\alpha,\beta}$ we denote the minimum of all $\nu_{\alpha,\beta}(G)$ for quotients $E \twoheadrightarrow G$ by $\nu^{\min}_{\alpha,\beta}(E)$.

**Lemma 4.5.** *Let* $E \twoheadrightarrow N$ *be an epimorphism in the category* $\mathcal{B}^{\alpha_0,\beta_0}$ *where* $N$ *is the semistable quotient in the Harder-Narasimhan filtration. Assume additionally that* $E$ *has no subobject with* $\nu_{\alpha_0,\beta_0} = \infty$*. Then there is an open subset* $U$ *around the point* $(\alpha_0,\beta_0)$ *such that the following holds.*

(i) *The inequality* $\nu^{\min}_{\alpha,\beta}(E) \leq \nu_{\alpha,\beta}(N)$ *holds for all* $(\alpha,\beta) \in U$.
(ii) *If* $N_1,\ldots,N_l$ *are the stable factors of* $N$*, then we obtain the inequality* $\nu^{\min}_{\alpha,\beta}(E) \geq \min \nu_{\alpha,\beta}(N_i)$ *for all* $(\alpha,\beta) \in U$.

*Proof.* By definition we have $\nu^{\min}_{\alpha_0,\beta_0}(E) = \nu_{\alpha_0,\beta_0}(N)$. Each semistable factor in the Harder-Narasimhan filtration of $E$ has a Jordan-Hölder filtration by stable factors. Since none of these stable factors has $\nu_{\alpha_0,\beta_0} = \infty$, we can use openness of stability ([BMT14, Corollary 3.3.3]) to show that all these stable factors are in the category $\mathcal{B}^{\alpha,\beta}$ in a small open neighborhood $U$ of $(\alpha_0, \beta_0)$. But that means $E$, $N$ and the kernel of $E \twoheadrightarrow N$ are in $\mathcal{B}^{\alpha,\beta}$ for all $\alpha, \beta \in U$. Therefore, we have $E \twoheadrightarrow N$ in $\mathcal{B}^{\alpha,\beta}$ for all $\alpha, \beta \in U$. But that implies $\nu^{\min}_{\alpha,\beta}(E) \leq \nu_{\alpha,\beta}(N)$ for all $(\alpha, \beta) \in U$.

We shrink $U$ such that the slopes of the $N_i$ are smaller than the slopes of all the other stable factors. We know that $E$ is an extension of all these stable factors. Therefore, it will be enough to show that whenever there is an exact sequence $0 \to A \to B \to C \to 0$ with $\nu^{\min}_{\alpha,\beta}(A), \nu^{\min}_{\alpha,\beta}(C) \geq a$, then $\nu^{\min}_{\alpha,\beta}(B) \geq a$ for any $a \in \mathbb{R}$.

Assume there is a semistable quotient $B \twoheadrightarrow D$ such that $\nu_{\alpha,\beta}(D) < a$. Due to $\nu^{\min}_{\alpha,\beta}(A) > \nu_{\alpha,\beta}(D)$ there is no morphism from $A$ to $D$. Therefore, $B \twoheadrightarrow D$ factors via a map $C \to D$. But there is also no non trivial map from $C$ to $D$ because of $\nu^{\min}_{\alpha,\beta}(C) > \nu_{\alpha,\beta}(D)$. But then $B \twoheadrightarrow D$ is trival which is a contradiction. □

This technical lemma allows to proceed with the proof of Theorem 4.1.

**Lemma 4.6.** *The inclusion $\mathcal{C} \subset \langle \mathcal{A}_{\alpha,\beta}, \mathcal{A}_{\alpha,\beta}[1] \rangle$ holds.*

*Proof.* If $L[i] \in \mathcal{B}^{\alpha,\beta}$ holds for a line bundle $L$ and $i \in \{0, 1\}$, then $L[i]$ is tilt-stable (see Proposition 7.4.1 in [BMT14]). By Lemma 4.3 we get immediately $\mathcal{O}(-1)[3], \mathcal{O}[1], \mathcal{O}(1) \in \langle \mathcal{A}_{\alpha,\beta}, \mathcal{A}_{\alpha,\beta}[1] \rangle$.

By [Ott88] the spinor bundle $S$ is $\mu$-stable. Since $\mu$-stability is preserved by the tensor product (see [HL10, Theorem 3.1.4]) we obtain $\mu$-stability of $S(-1)$. The inequality $\mu_{\alpha,\beta}(S(-1)) \leq 0$ leads to $S(-1)[1] \in \mathcal{B}^{\alpha,\beta}$. In order to show $S(-1)[1] \in \mathcal{A}_{\alpha,\beta}$ we need to prove that any quotient $S(-1)[1] \twoheadrightarrow G$ in $\mathcal{B}^{\alpha,\beta}$ satisfies $\nu_{\alpha,\beta}(G) > 0$. The proof proceeds in three steps. At first we show $S(-1)[1]$ has no proper subobject of slope $\infty$. Then we prove stability of $S(-1)[1]$ for $\beta = 0$. Finally, we use the previous lemma to reduce to this case.

Assume we have a proper subobject $A \hookrightarrow S(-1)[1]$ with $\nu_{\alpha,\beta}(A) = \infty$. That means $\mathrm{ch}^{\beta}_1(A) = 0$ and moreover $\mathrm{ch}^{\beta}_1(H^{-1}(A)) = 0$. Suppose we have $H^{-1}(A) \neq 0$. Then the injective morphism $H^{-1}(A) \hookrightarrow S(-1)$ in $\mathrm{Coh}(Q)$ constitutes a contradiction to the $\mu_{\alpha,\beta}$-stability of $S(-1)$ with the inequality $\mu_{\alpha,\beta}(S(-1)) \leq 0$. Hence, $H^{-1}(A) = 0$ and since $\mathrm{ch}^{\beta}_1(A) = 0$, it follows that $A$ has rank 0 and is supported in dimension less than or equal to one. But in that case Serre duality implies $\mathrm{Hom}(A, S(-1)[1]) = 0$ which is a contradiction to $A \to S(-1)[1]$ being a monomorphism.

Assume we have an exact sequence $0 \to A \to S(-1)[1] \to G \to 0$ in $\mathcal{B}^{\alpha,0}$ with $\nu_{\alpha,0}(G) \leq \nu_{\alpha,0}(A)$. The long exact sequence in cohomology implies that $G \simeq N[1]$ for $N \in \mathrm{Coh}\, Q$. Since $\mathrm{ch}_1(S(-1)[1]) = H$ (see Lemma 4.3) and $\nu_{\alpha,0}(G) \neq \infty$, we obtain $\mathrm{ch}_1(G) = H$ and $\mathrm{ch}_1(A) = 0$. But then $\nu_{\alpha,0}(A) = \infty$, a case that we had already ruled out.

Assume there is $\alpha_0 \in (0, 1/3)$ and $\beta_0 \in [-1/2, 0)$ such that the inequality $\nu^{\min}_{\alpha_0,\beta_0}(S(-1)[1]) \leq 0$ holds. Since stability is an open property by [BMT14, Corollary 3.3.3] and $S(-1)[1]$ is $\nu_{\alpha_0,0}$-stable, we get

$$\beta_1 := \sup\{\beta \leq 0 : \nu^{\min}_{\alpha_0,\beta}(S(-1)[1]) \leq 0\} < 0.$$

Let $S(-1)[1] \twoheadrightarrow N$ be a semistable quotient in $\mathcal{B}^{\alpha_0,\beta_1}$ as in Lemma 4.5. Assume $\nu_{\alpha_0,\beta_1}(N) > 0$ and let $N_1, \ldots, N_l$ be the stable quotients in the Jordan-Hölder filtration of $N$. In a neighborhood around $(\alpha_0, \beta_1)$ we have the inequality $\nu^{\min}_{\alpha,\beta}(S(-1)[1]) \geq \min \nu_{\alpha,\beta}(N_i) > 0$, which is a contradiction to the choice of $\beta_1$. Therefore, we know $\nu_{\alpha_0,\beta_1}(N) \leq 0$. We define the function

$$\begin{aligned} f(\beta) &= \frac{\alpha_0^2 H^2 \operatorname{ch}_1^{\beta_1}(N) \nu_{\alpha_0,\beta}(N)}{\alpha_0} \\ &= H \operatorname{ch}_2(N) - \beta H^2 \operatorname{ch}_1(N) + \frac{\beta^2 H^3}{2} \operatorname{ch}_0(N) - \frac{\alpha_0^2 H^3}{2} \operatorname{ch}_0(N). \end{aligned}$$

We have the inequalities $f(\beta_1) \leq 0$ and $f'(\beta_1) = -H^2 \operatorname{ch}_1^{\beta_1}(N) < 0$. As $\nu^{\min}_{\alpha_0,\beta}(S(-1)[1]) \leq \nu_{\alpha_0,\beta}(N)$ in a neighborhood of $(\alpha_0, \beta_1)$, the fact that $f$ is decreasing at $\beta_1$ is a contradiction to the choice of $\beta_1$. □

The proof of Theorem 4.1 can be concluded by the next lemma.

**Lemma 4.7.** *For all $x \in X$, we have $k(x) \in \mathcal{C}$ and for all proper subobjects $C \hookrightarrow k(x)$ in $\mathcal{C}$ the inequality $\Im Z_{\alpha,\beta,\frac{1}{6}}(C) > 0$ holds.*

*Proof.* We have $k(x) \in \mathcal{C}$ because of the resolution in (3)

$$0 \to \mathcal{O}(-1) \to S(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{O}(1) \to k(x) \to 0.$$

For the second assertion we need to figure out which are the subobjects of $k(x) \in \mathcal{C}$. Any object in $\mathcal{C}$ is given by a complex $F$ of the form

$$0 \to \mathcal{O}(-1)^{\oplus a} \to S(-1)^{\oplus b} \to \mathcal{O}^{\oplus c} \to \mathcal{O}(1)^{\oplus d} \to 0.$$

for $a, b, c, d \in \mathbb{Z}_{\geq 0}$. Since $\mathcal{C}$ is the category of representations of a quiver with relations with simple objects $\mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1)$, we can interpret $v(F) = (a, b, c, d)$ as the dimension vector of that representation. Therefore, $F \hookrightarrow k(x) \twoheadrightarrow G$ implies $a \leq 1$, $b \leq 2$, $c \leq 4$ and $d \leq 1$. If $F$ is non trivial, then there is a simple object $T_1 \hookrightarrow F$. But the only simple object with non trivial morphism into $k(x)$ is $\mathcal{O}(1)$. Therefore, the equality $d = 1$ holds. If $G$ is non trivial, then there exists a simple quotient $k(x) \twoheadrightarrow T_2$. By Serre duality, the only simple quotient is $T_2 = \mathcal{O}(-1)[3]$. That implies $a = 0$. Assume $b = 2$, but $c < 4$. Then we obtain $G = \mathcal{O}(-1)[3] \oplus \mathcal{O}[1]^{\oplus 4-c} \twoheadrightarrow \mathcal{O}[1]$. A contradiction comes from $\operatorname{Hom}(k(x), \mathcal{O}[1]) = 0$. Therefore, $b = 2$ implies $c = 4$. The remaining cases are

$$v(F) \in \{(0, 2, 4, 1)\} \cup \{(0, b, c, 1) : b \in \{0, 1\}, c \in \{0, 1, 2, 3, 4\}\}.$$

Since $\Im Z_{\alpha,\beta,\frac{1}{6}}(S(-1)) < 0$, the case $b = 0$ will follow from $b = 1$. With the same argument $v(F) = (0, 1, 4, 1)$ will follow from $v(F) = (0, 2, 4, 1)$. Depending on the sign of $\Im Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O})$, we can reduce the situation with $b = 1$ to either $c = 0$ or $c = 4$. Hence, we are left to check two cases.

| $v(F)$ | $\Im Z_{\alpha,\beta,\frac{1}{6}}$ |
|---|---|
| $(0, 2, 4, 1)$ | $\alpha((1+\beta)^2 - \alpha^2)$ |
| $(0, 1, 0, 1)$ | $\alpha(1 - 3(\beta^2 - \alpha^2))$ |

For all of them $\Im Z_{\alpha,\beta,\frac{1}{6}}$ is positive. □

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