

Stability Conditions on Threefolds - First Wall Crossings

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Why care about stability?

Birational geometry of moduli of sheaves on surfaces, e.g. Hilbert schemes of points.

- \mathbb{P}^2 : Arcara-Bertram-Coskun-Huizenga-Woolf.
- Abelian surfaces: Maciocia-Meachan, Minamide-Yanagida-Yoshioka.
- K3 surfaces: Bayer-Macrì.
- Enriques surfaces: Nuer.

Wall-crossing formula for Donaldson-Thomas Invariants for Bridgeland stability by Kontsevich and Soibelman. Joyce and Song had obtained this for Gieseker stability.

Bayer, Bertram, Macrì and Toda proved the following theorem about Fujita's Conjecture.

Theorem: Let X be any smooth projective threefold such that the Conjecture on the right holds and let L be an ample line bundle. Then $\omega_X \otimes L^{\otimes 6}$ is very ample.

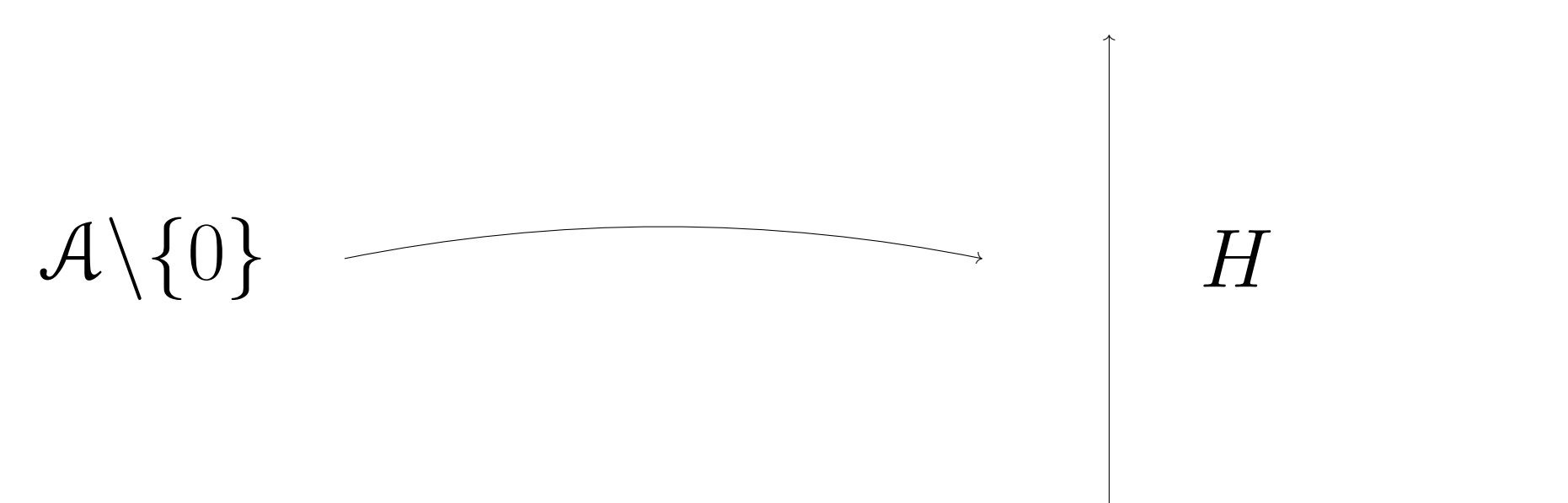
References

- [Sch13] Schmidt, B.: A generalized Bogomolov-Gieseker inequality for the smooth quadric threefold. Bull. Lond. Math. Soc. 46 (2014), no. 5, 915-923.
- [Sch15] Schmidt, B.: Bridgeland Stability on Threefolds - First Wall Crossings, in preparation.
- [GHS15] Gallardo, P.; Lozano Huerta, C.; Schmidt, B.: On the Hilbert Scheme of Elliptic Quartics, in preparation.

Definition

Let H be the upper half plane plus the negative real line and X a smooth projective threefold. Bridgeland stability condition (Z, \mathcal{A}) on $D^b(X)$:

- \mathcal{A} is the heart of a bounded t-structure.
- $Z : K_0(X) = K_0(\mathcal{A}) \rightarrow \mathbb{C}$ is a homomorphism.
- $Z(\mathcal{A} \setminus \{0\}) \subset H$ and some technical properties.



Questions & Goals

Let X be a smooth projective threefold over \mathbb{C} . Let $E \in \text{Coh}(X)$ and $v = \text{ch}(E)$.

- What are the moduli spaces of Bridgeland stable complexes with fixed Chern character v ?
- Are all coarse moduli spaces of Bridgeland stable complexes with fixed Chern character v projective?
- Do Bridgeland stability conditions even exist on smooth projective threefolds?
- Let C be a curve in \mathbb{P}^3 and $v = \text{ch}(\mathcal{I}_C)$. What is the relation between moduli of Bridgeland stable complexes with Chern character v and the Hilbert schemes of curves containing C ?

Tilting/Details

Twisted Chern character: If B is any \mathbb{R} -divisor, then ch^B is defined to be $e^{-B} \text{ch}$.

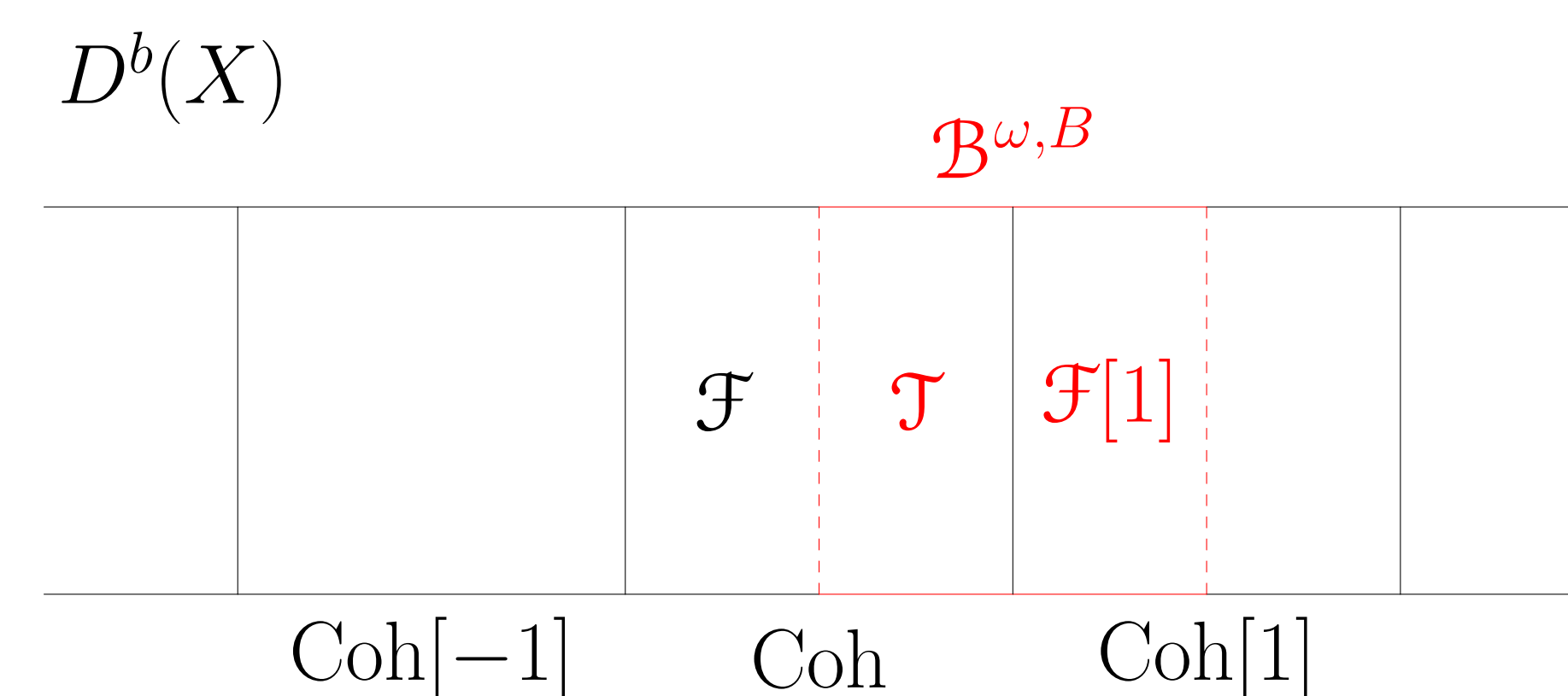
Tilting: Let ω be any ample \mathbb{R} -divisor.

$$\mu_{\omega, B} = \frac{\omega^2 \text{ch}_1^B}{\omega^3 \text{ch}_0^B}$$

$$\mathcal{T}_{\omega, B} = \{E \in \text{Coh}(X) : \forall E \twoheadrightarrow G, \mu_{\omega, B}(G) > 0\}$$

$$\mathcal{F}_{\omega, B} = \{E \in \text{Coh}(X) : \forall F \hookrightarrow E, \mu_{\omega, B}(F) \leq 0\}$$

The tilted abelian category is defined by the extension closure $\mathcal{B}^{\omega, B} = \langle \mathcal{F}_{\omega, B}[1], \mathcal{T}_{\omega, B} \rangle$ consisting of some two term complexes.



Tilt Stability: A new slope function on $\mathcal{B}^{\omega, B}$ is defined by

$$\nu_{\omega, B} := \frac{\omega \text{ch}_2^B - \frac{\omega^3}{2} \text{ch}_0^B}{\omega^2 \text{ch}_1^B}$$

This gives Bridgeland stability conditions on surfaces, but it is not enough on threefolds.

Bridgeland Stability

An analogous tilt of $\mathcal{B}^{\omega, B}$ leads to a category of three term complexes $\mathcal{A}^{\omega, B}$. We define

$$Z_{\omega, B, s} = (-\text{ch}_3^B + s\omega^2 \text{ch}_1^B) + i(\omega \text{ch}_2^B - \frac{\omega^3}{2} \text{ch}_0^B)$$

with slope function

$$\lambda_{\omega, B, s} = \frac{\Re(Z_{\omega, B, s})}{\Im(Z_{\omega, B, s})}$$

Conjecture

- X smooth projective threefold.
- ω any ample \mathbb{R} -divisor.
- B any \mathbb{R} -divisor.

Then $(Z_{\omega, B, s}, \mathcal{A}^{\omega, B})$ is a Bridgeland stability condition for any $s > \frac{1}{6}$.

BMST Inequality

The conjecture is equivalent to the fact that any $\nu_{\omega, B}$ -stable object $E \in \mathcal{B}^{\omega, B}$ satisfies the inequality (Bayer-Macrì-Stellari-Toda)

$$\Delta(E) + 4(\omega \text{ch}_2^B(E))^2 - 6(\omega^2 \text{ch}_1^B(E)) \text{ch}_3^B(E) \geq 0,$$

where

$$\Delta(E) = (\omega^2 \text{ch}_1^B(E))^2 - 2(\omega^3 \text{ch}_0^B(E))(\omega \text{ch}_2^B(E)).$$

Theorem (Macrì, Schmidt)

Let X be \mathbb{P}^3 or the quadric threefold.

- The conjecture holds.
- If $1 \gg \omega > 0$, then all coarse moduli spaces of Bridgeland semistable objects are moduli of quiver representations. In particular, they are projective.

The conjecture is also known for Abelian threefolds due to Maciocia-Piyaratne and independently Bayer-Macrì-Stellari. It is unknown whether the moduli spaces are projective. However, Piyaratne and Toda showed that they are proper and of finite type over \mathbb{C} whenever the conjecture holds.

The Projective Space

Let $v = i \text{ch}(\mathcal{O}_{\mathbb{P}^3}(m)) - j \text{ch}(\mathcal{O}_{\mathbb{P}^3}(n))$ for positive integers i, j and integers m, n with $n < m$. Assume further that (v_0, v_1, v_2) is a primitive vector. Let W be the locus where $\lambda_{\omega, B, s}(\mathcal{O}(m)) = \lambda_{\omega, B, s}(\mathcal{O}(n))$.

Theorem (The Last Wall)

There is a path $\gamma : [0, 1] \rightarrow \text{Stab}(\mathbb{P}^3)$ ending inside the area enclosed by W that satisfies the following properties.

- The locus W is the last wall on γ for the class v . After the wall the moduli space of semistable objects is empty. Before the wall, the moduli space is smooth and irreducible.
- At the beginning of the path the moduli space is the same as the moduli space of Gieseker stable coherent sheaves E with $\text{ch}(E) = v$.

Elliptic Quartic Curves

The Chern character of the ideal sheaf of an elliptic quartic curve in \mathbb{P}^3 is given by

$$v = (1, 0, -4, 8) = 2 \text{ch}(\mathcal{O}_{\mathbb{P}^3}(-2)) - \text{ch}(\mathcal{O}_{\mathbb{P}^3}(-4)).$$

A complete list of wall crossings on a path close to the left of the hyperbola $\Im Z_{\omega, B, s}(1, 0, -4, 8) = 0$:

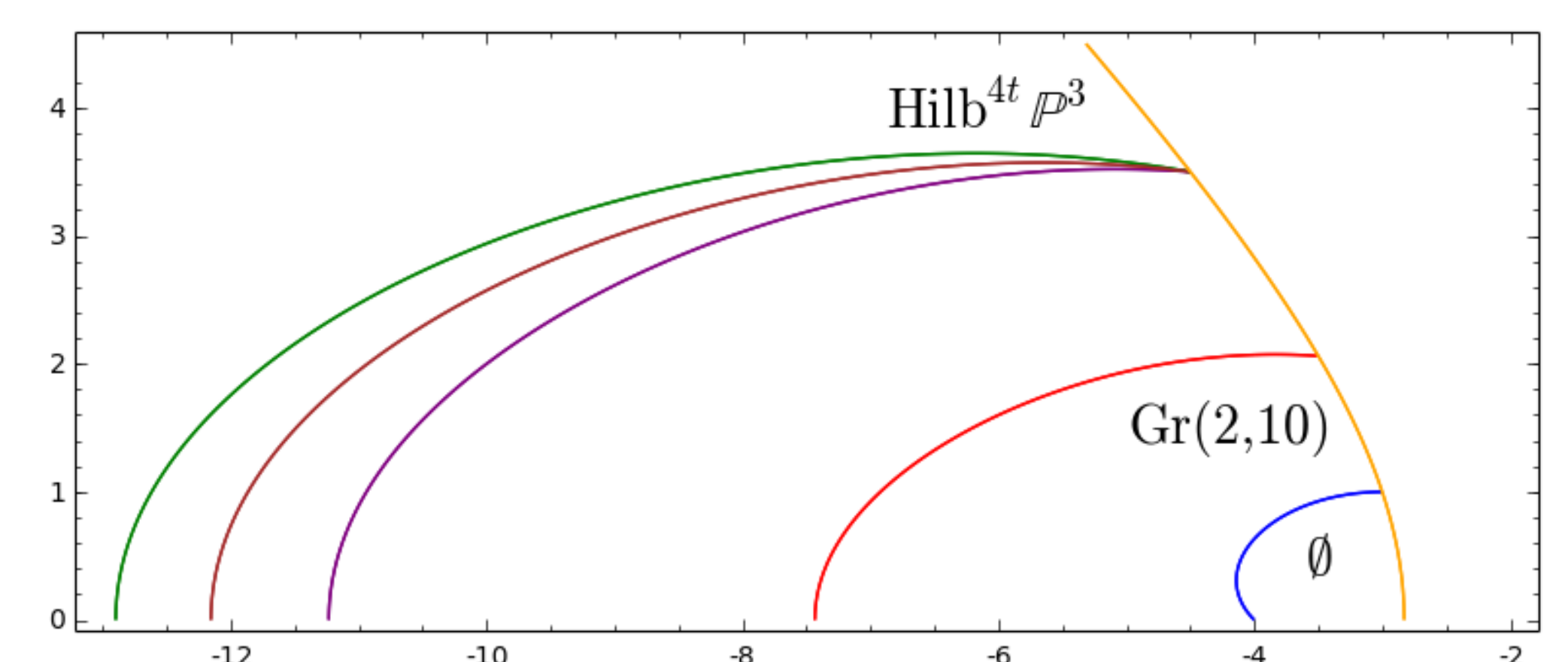


Figure : Walls in Bridgeland stability

All strictly semistable objects at each of the walls are extensions of two objects.

- $\mathcal{O}(-2)^{\oplus 2}, \mathcal{O}(-4)[1]$, the wall W .
- $\mathcal{I}_L(-1), \mathcal{O}_V(-3)$, $L \subset \mathbb{P}^3$ line, $V \subset \mathbb{P}^3$ plane.
- $\mathcal{I}_Z(-1), \mathcal{O}_V(-4)$ $Z \subset \mathbb{P}^3$ length two subscheme.
- $\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)$, $P \in \mathbb{P}^3$, $Q \in V$.
- $\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)$ $Z' \subset V$ length two subscheme.