Generalized Bogomolov-Gieseker Inequality for the Quadric Threefold

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Basic Object

Bridgeland introduced Bridgeland stability as an analogue of slope stability in a triangulated category (see [Bri07]). Instead of defining it in $\mathrm{Coh}(X)$, one uses other abelian categories inside $D^b(X)$ with a different slope. The set of Bridgeland stability conditions has the structure of a complex manifold.

For curves a sheaf of rank 0 has positive degree. Is there a higher dimensional analogue of this?

How to construct it?

Surfaces: Was constructed by Arcara, Bertram and Bridgeland using:

Bogomolov-Gieseker inequality
- (X, H) smooth projective polarized surface
- F slope semistable sheaf with respect to H

Then

$$\mathrm{ch}_1(F)^2 - 2\,\mathrm{ch}_2(F)\,\mathrm{ch}_0(F) \geq 0.$$

Threefolds: More complicated. Bayer, Macrì and Toda proposed a conjectural construction in [BMT11] that we will confirm for the smooth projective quadric threefold.

Applications

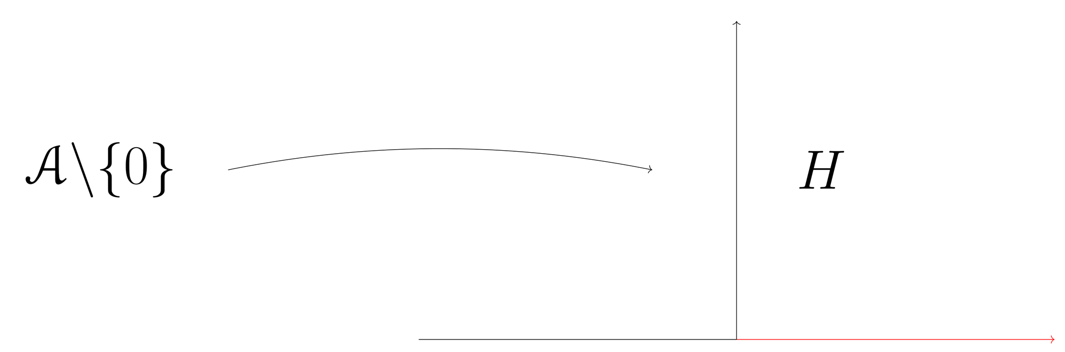
By looking at moduli spaces of stable complexes one can analyse the birational geometry of the Hilbert scheme of points on a surface.
- Abelian surfaces by Maciocia, Meachan, Yanagida and Yoshioka.
- K3 surfaces by Bayer and Macrì.
- $\mathbb{P}^2$ by Arcara, Bertram, Coskun and Huizenga.

Kontsevich and Soibelman did work on a wall-crossing formula for Donaldson-Thomas Invariants on a Calabi-Yau threefold for Bridgeland stability. Previously, Joyce and Song had described such a formula for Gieseker stability.

Bayer, Bertram, Macrì and Toda proved the following theorem about Fujita's Conjecture.
Theorem: Let X be any smooth projective threefold and L be an ample line bundle such that the Conjecture on the right holds. Then $\omega_X \otimes L^{\otimes 6}$ is very ample.

Definition

Let $H := \{re^{i\pi\varphi} : r > 0, \varphi \in (0,1]\}$ be the upper half plane plus the negative real line.
Bridgeland stability condition $(Z, \mathcal{A})$ on $D^b(X)$:
- $\mathcal{A}$ is the heart of a bounded t-structure.
- $Z : K_0(X) = K_0(\mathcal{A}) \to \mathbb{C}$ is a homomorphism.
- $Z(\mathcal{A}\backslash\{0\}) \subset H$.



- Some technical property.

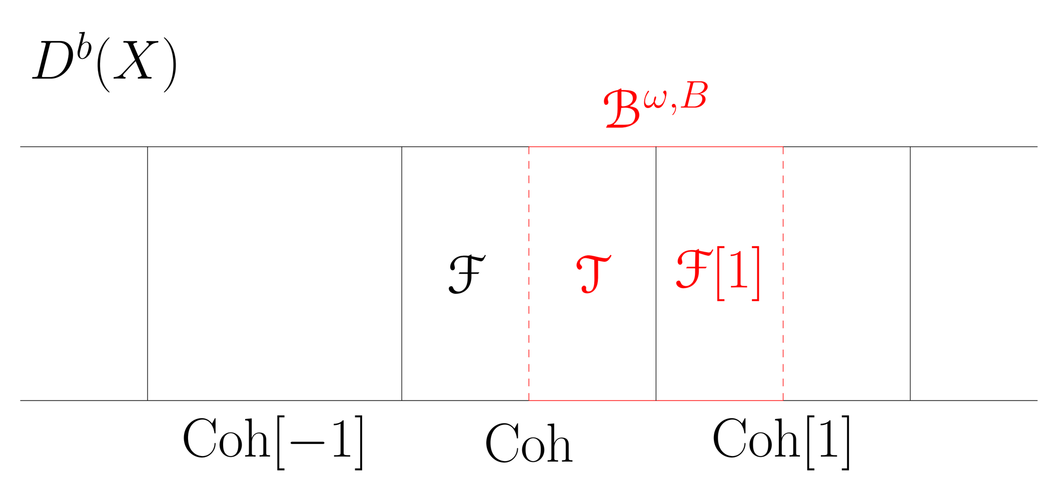
Tilting/Details

Twisted Chern character: If B is any $\mathbb{R}$-divisor, then ch^B is defined to be $e^{-B}\,\mathrm{ch}$.

Tilting: Let ω be any ample $\mathbb{R}$-divisor.

$$\mu_{\omega,B} = \frac{\omega^2\,\mathrm{ch}_1^B}{\omega^3\,\mathrm{ch}_0^B}$$
$$\mathcal{T}_{\omega,B} = \{E \in \mathrm{Coh}(X) : \forall E \twoheadrightarrow G, \mu_{\omega,B}(G) > 0\}$$
$$\mathcal{F}_{\omega,B} = \{E \in \mathrm{Coh}(X) : \forall F \hookrightarrow E, \mu_{\omega,B}(F) \leq 0\}$$

The tilted abelian category is defined by the extension closure $\mathcal{B}^{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B}\rangle$ consisting of some two term complexes.



Tilt Stability: A new slope function on $\mathcal{B}^{\omega,B}$ is defined by

$$\nu_{\omega,B} := \frac{\omega\,\mathrm{ch}_2^B - \frac{\omega^3}{6}\mathrm{ch}_0^B}{\omega^2\,\mathrm{ch}_1^B}.$$

An object $E \in \mathcal{B}^{\omega,B}$ is called tilt-stable if for any exact sequence $0 \to F \to E \to G \to 0$ the inequality $\nu_{\omega,B}(F) < \nu_{\omega,B}(G)$ holds.

Conjecture

- X smooth projective threefold
- ω any ample $\mathbb{R}$-divisor
- B any $\mathbb{R}$-divisor
- $E \in \mathcal{B}^{\omega,B}$ is $\nu_{\omega,B}$-stable with $\nu_{\omega,B}(E) = 0$

Then

$$\mathrm{ch}_3^B(E) \leq \frac{\omega^2}{18}\mathrm{ch}_1^B(E).$$

Known Cases
- $\mathbb{P}^3$ due to Macrì.
- Principally polarized abelian threefolds of Picard rank one due to Maciocia and Piyaratne.

Bridgeland Stability

An analogous tilt of $\mathcal{B}^{\omega,B}$ leads to a category of three term complexes $\mathcal{A}^{\omega,B}$. Assuming the conjecture is true we obtain a Bridgeland stability condition by

$$Z_{\omega,B} := (-\mathrm{ch}_3^B + \frac{\omega^2}{2}\mathrm{ch}_1^B) + i(\omega\,\mathrm{ch}_2^B - \frac{\omega^3}{6}\mathrm{ch}_0^B).$$

with slope

$$\lambda_{\omega,B} := -\frac{\Re(Z_{\omega,B})}{\Im(Z_{\omega,B})}.$$

Main result

Assumptions
- Z is the smooth projective quadric threefold
- ω any ample $\mathbb{R}$-divisor
- B any $\mathbb{R}$-divisor
- $E \in \mathcal{B}^{\omega,B}$ is $\nu_{\omega,B}$-stable with $\nu_{\omega,B}(E) = 0$

Conclusion

$$\mathrm{ch}_3^B(E) \leq \frac{\omega^2}{18}\mathrm{ch}_1^B(E)$$

In particular, a large family of Bridgeland stability conditions can be constructed on Z.

Future Plans

I am currently studying the Conjecture on toric threefolds. In particular, I investigate the behaviour under birational transformations.

Quiver Representations

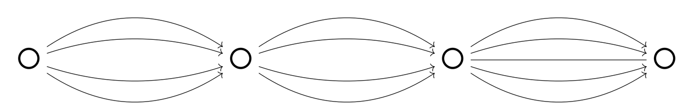
The spinor bundle S is defined via an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \to \iota_* S \to 0$$

where $\iota : Z \hookrightarrow \mathbb{P}^4$ is a closed embedding. Due to Kapranov

$$\mathcal{O}(-1), S(-1), \mathcal{O}, \mathcal{O}(1)$$

is a strong full exceptional collection on $D^b(Z)$. Together with a result from Bondal $D^b(Z)$ is equivalent to the derived category of the following quiver Q with some relations.

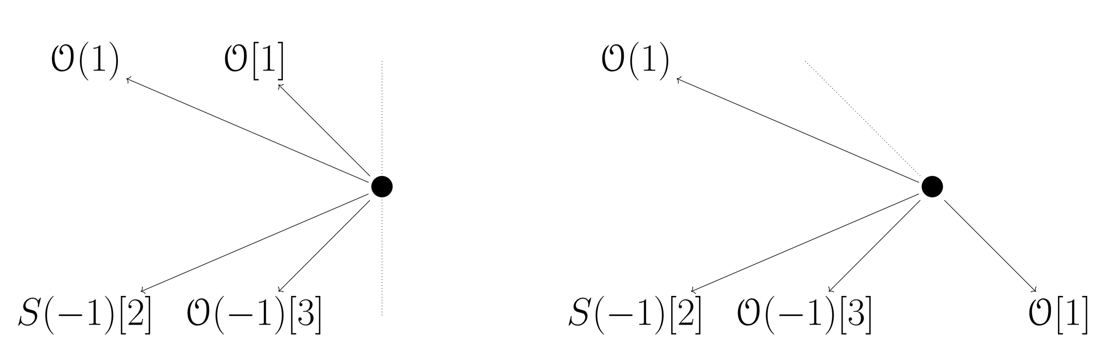


Under this equivalence we have

$$\mathrm{Rep}(Q) \cong \langle \mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1)\rangle.$$

Idea of the Proof

- Let $\mathcal{O}(H) \cong \mathcal{O}(1)$ and $B = \beta H$, $\omega = \alpha H$.
- Reduce to $0 < \alpha < 1/3$ and $-1/2 \leq \beta \leq 0$.
- Compute values of $\mathrm{Rep}(Q)$ under $Z_{\omega,B}$.



$|\beta| \leq |\alpha|$ $\qquad$ $|\beta| > |\alpha|$

- Show that $\mathcal{A}^{\omega,B}$ is a tilt of $\mathrm{Rep}(Q)$. Then the inequality is a consequence of $\mathrm{Rep}(Q)$ mapping into a half-plane.

References

[BMT11] Bayer, A.; Macrì, E.; Toda, Y.: Bridgeland Stability conditions on threefolds I: Bogomolov-Gieseker type inequalities, 2011. arXiv:1103.5010

[Bri07] Bridgeland, T.: Stability conditions on triangulated categories. Ann. of Math. (2) 166 (2007), no. 2, 317-345.

[Sch13] Schmidt, B.: A generalized Bogomolov-Gieseker inequality for the smooth quadric threefold, 2013. arXiv:1309.4265