

Stability Conditions on Threefolds and Space Curves

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Abstract

This thesis investigates both constructions and applications of Bridgeland stability conditions on smooth complex projective varieties of dimension three.

A conjectural construction of stability condition on threefolds due to Bayer, Macrì and Toda will be proven for the smooth quadric threefold and analogies to the proof for three-dimensional projective space will be pointed out. This implies the existence of a large family of Bridgeland stability conditions. Moreover, we will give a counterexample to the conjecture for the blow up of three dimensional projective space in a point.

In between slope stability and Bridgeland stability there is the notion of tilt stability. Computations in it are similar to those for Bridgeland stability on surfaces. A technique to translate computations in tilt stability to wall crossings in Bridgeland stability will be developed.

The author computes first examples of wall crossing behaviour in three dimensional projective space. In particular, for Hilbert schemes of curves such as twisted cubics or complete intersection curves of the same degree, two chambers in the stability manifold are described where the moduli space is given by a smooth projective irreducible variety respectively the Hilbert scheme. In the cases of twisted cubics and elliptic quartics, all walls and moduli spaces on a path between those two chambers are computed. This recovers former results about the geometry of these spaces.

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On the birational geometry of Schubert varieties, Bull. Soc. Math. France 143 (2015), no. 3, 489–502.

Counterexample to the generalized Bogomolov-Gieseker inequality for threefolds, 2016. Preprint.

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Chapter 1: Introduction

The central notion in complex algebraic geometry is that of a smooth complex projective variety. It is a closed complex submanifold of complex projective space. The idea of a moduli space is key to many modern developments in algebraic geometry. Instead of studying one object at a time it is often convenient to study many of them at the same time. A moduli space is a space in which every point corresponds to an object. In this way we can study these families of objects by geometry itself. For example one can regard collections of n points on a given variety. All these collections can be studied at the same time by making them into a space called the Hilbert scheme of n points. In this thesis, we further develop the study of moduli spaces of complexes in the sense of Bridgeland and apply it to study Hilbert schemes parametrizing curves in three dimensional projective space.

1.1 Existence of Stability Conditions

Let X be a smooth projective variety over \mathbb{C} of dimension n . The main homological invariant we use is the bounded derived category of coherent sheaves $D^b(X)$. It was originally defined by Verdier in his thesis with Grothendieck [Ver96] as a book-keeping tool for cohomology calculations. Over the years it turned out to be an interesting object in itself.

Bridgeland introduced the notion of a stability condition (see Definition 2.2.16) in the bounded derived category of coherent sheaves $D^b(X)$ as an analogue of classical slope stability. For slope stability with respect to a given polarization H one defines a number

$$\mu_H(E) = \frac{H^{n-1} \text{ch}_1(E)}{H^n \text{ch}_0(E)}$$

called the *slope* for any coherent sheaf $E \in \text{Coh}(X)$. A coherent sheaf is then called *slope semistable* if all proper non trivial subsheaves have smaller slope. For Bridgeland stability one replaces the category of coherent sheaves by a different abelian subcategory $\mathcal{A} \subset D^b(X)$ and replaces the slope by a homomorphism $Z : K_0(X) \rightarrow \mathbb{C}$, where $K_0(X)$ is the Grothendieck group. The slope is then given by

$$\mu(E) = \frac{\Re Z(E)}{\Im Z(E)}$$

for any $E \in \mathcal{A}$. As a further technical property one demands that every object in $D^b(X)$ has a canonical filtration into semistable factors called the *Harder-Narasimhan filtration*. The homomorphism Z is usually called the *central charge* of the stability condition.

Bridgeland's main motivation was work by Douglas [Dou02] in string theory. The interests of string theorists lie in Calabi-Yau threefolds. While many other applications have emerged from Bridgeland's work, the construction of even a single stability condition on a simply-connected Calabi-Yau threefold has eluded algebraic geometers till this day.

In the case of surfaces the construction was settled in [Bri08, AB13]. For any ample divisor H and real numbers $\alpha > 0$ and $\beta \in \mathbb{R}$ there is a stability condition. The construction is based on the classical Bogomolov inequality. It says that any

semistable sheaf E on a smooth projective complex surface satisfies an inequality involving Chern characters given by

$$\mathrm{ch}_1(E)^2 - 2 \mathrm{ch}_0(E) \mathrm{ch}_2(E) \geq 0.$$

It is possible to do the same construction on a smooth projective threefold, but one does not obtain a Bridgeland stability condition. In [BMT14] this is called tilt stability (see beginning of Subsection 2.2.2), and they conjecture an inequality for tilt semistable objects that would allow the construction of Bridgeland stability by repeating the process with this new notion of stability. Over time their conjecture has evolved into the following form.

Conjecture 1.1.1 (BMT-Inequality, [BMT14, BMS14, PT15]). *Let X be a smooth projective complex threefold with polarization H . Then any tilt semistable object E with respect to H , $\alpha \in \mathbb{R}_{>0}$, and $\beta \in \mathbb{R}$ satisfies the inequality*

$$\begin{aligned} Q_{\alpha,\beta}(E) = & \alpha^2((H^2 \cdot \mathrm{ch}_1^\beta(E))^2 - (H^3 \cdot \mathrm{ch}_0^\beta(E))(2H \cdot \mathrm{ch}_2^\beta(E))) \\ & + (2H \cdot \mathrm{ch}_2^\beta(E))^2 - 6(H^2 \cdot \mathrm{ch}_1^\beta(E)) \mathrm{ch}_3^\beta(E) \geq 0, \end{aligned}$$

where $\mathrm{ch}^\beta = e^{-\beta H} \cdot \mathrm{ch}$.

We will prove the following result.

Theorem 1.1.2. [Sch14] *Let $Q \subset \mathbb{P}^4$ be the smooth quadric threefold cut out by an equation of degree two. The BMT-inequality (Conjecture 1.1.1) holds for Q .*

For some varieties there is an equivalence between the bounded derived category of coherent sheaves and the bounded derived category of representations of a quiver with relations due to Bondal [Bon89]. This phenomena was first observed by Beilinson

[Bei78] in the case of projective space and provides a surprising connection between algebraic geometry and representation theory. We will use this connection repeatedly in the technical details of proofs. In particular, it is a crucial ingredient to the proof of the previous theorem. For Q the existence of such an equivalence has been established in [Kap88].

Unfortunately, it turns out that the BMT-inequality does not hold for general X . We give a counterexample when X is the blow up of \mathbb{P}^3 in one point. Let L be the pullback of the very ample generator of $\text{Pic}(\mathbb{P}^3)$ and let E be the exceptional divisor of $X \rightarrow \mathbb{P}^3$. The variety X is Fano with canonical divisor $\omega = -4L + 2E$. In particular, the divisor $H = 2L - E$ is ample.

Theorem 1.1.3 ([Sch16]). *There exists $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that the line bundle $\mathcal{O}_X(L)$ is tilt stable, but the BMT inequality fails.*

1.2 Moduli Spaces of Semistable Objects

A Bridgeland stability condition, together with a fixed set of numerical invariants for semistable objects, allows one to construct a moduli space whose points are in correspondence with semistable objects of those numerical invariants. The remarkable idea of Bridgeland is that there are different stability conditions that together form a manifold, called the stability manifold $\text{Stab}(\mathbb{P}^3)$. More precisely, this means that after deforming the central charge Z slightly, one can still find a category \mathcal{A} to get a stability condition. Moreover, there is a locally finite wall and chamber structure on this manifold such that moduli spaces for fixed numerical invariants are constant outside of walls. The Hilbert scheme can usually be identified with some moduli space

coming from Bridgeland stability. Varying within the stability manifold will lead to other related spaces.

A central problem that motivated this thesis is the connection between moduli spaces of complexes and Hilbert schemes, and how it can be used to study concrete problems in classical complex algebraic geometry. We investigate the possibility of studying moduli spaces of complexes in projective space \mathbb{P}^3 . The following theorem applies to the special cases of complete intersections of the same degree and to twisted cubics.

Theorem 1.2.1 (See also Theorem 4.3.3). *Let $v = i \operatorname{ch}(\mathcal{O}_{\mathbb{P}^3}(m)) - j \operatorname{ch}(\mathcal{O}_{\mathbb{P}^3}(n))$ where $m, n \in \mathbb{Z}$ are integers with $n < m$ and $i, j \in \mathbb{N}$ are positive integers. Assume that (v_0, v_1, v_2) is a primitive vector. There is a path $\gamma : [0, 1] \rightarrow \operatorname{Stab}(\mathbb{P}^3)$ that satisfies the following properties.*

- (i) *At $\gamma(1)$ the semistable objects are exactly slope stable coherent sheaves E with $\operatorname{ch}(E) = v$.*
- (ii) *At $\gamma(0)$ there are no semistable objects, i.e. the moduli space is empty.*
- (iii) *Let $c \in (0, 1)$ be such that the moduli space of semistable objects is empty at $\gamma(t)$ for $t \in [0, c)$. Then for small enough $\varepsilon > 0$ the moduli space of semistable objects at $\gamma(c + \varepsilon)$ is smooth, irreducible and projective.*

The key statement in this Theorem is the fact that we can obtain a smooth, irreducible and projective variety. Vakil proved Murphy's Law for Hilbert schemes of curves in projective space in [Vak06]. It says these spaces can have arbitrarily bad singularities and are generally not irreducible. As an application, we compute all walls on the path of the last Theorem in the case of twisted cubics.

Theorem 1.2.2 (See also Theorem 5.1.3). *Let $v = (1, 0, -3, 5) = \text{ch}(\mathcal{I}_C)$ where $C \subset \mathbb{P}^3$ is a twisted cubic curve. There is a path $\gamma : [0, 1] \rightarrow \text{Stab}(\mathbb{P}^3)$ such that the moduli spaces for v in its image outside of walls are given in the following order.*

- (i) *The empty space $M_0 = \emptyset$.*
- (ii) *A smooth projective variety M_1 that contains ideal sheaves of twisted cubic curves as an open subset.*
- (iii) *A reducible variety with two components $M_2 \cup M'_2$. The space M_2 is a blow up of M_1 in a smooth locus isomorphic to the incidence variety of a point contained in a plane in \mathbb{P}^3 . The exceptional locus parametrizes plane singular cubic curves with a spatial embedded point at a singularity. The second component M'_2 is a \mathbb{P}^9 -bundle over $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$. An open subset in M'_2 parametrizes plane cubic curves with a point not contained in the plane.*
- (iv) *The Hilbert scheme of curves C with $\text{ch}(\mathcal{I}_C) = (1, 0, -3, 5)$. It is given as $M_2 \cup M'_3$ where M'_3 is a blow up of M'_2 in a smooth locus isomorphic to the incidence variety of a point contained in a plane in \mathbb{P}^3 . The exceptional locus parametrizes plane cubic curves together with a point scheme theoretically contained in the plane.*

The Hilbert scheme of twisted cubics has been heavily studied. In [PS85] it was shown that it has two smooth irreducible components of dimension 12 and 15 intersecting transversally in a locus of dimension 11. In [EPS87] it was shown that the closure of the space of twisted cubics in this Hilbert scheme is the blow up of another smooth projective variety in a smooth locus. This matches exactly the description we obtain using stability.

As a second example, we deal with the situation for elliptic quartics in a similar manner. We prove the following Theorem that also reproves a result by Vainsencher and Avritzer in [VA92] describing the main component of this Hilbert scheme.

Theorem 1.2.3. *Let $v = (1, 0, -4, 8) = \text{ch}(\mathcal{I}_C)$ where $C \subset \mathbb{P}^3$ is an elliptic quartic curve. There is a path $\gamma : [0, 1] \rightarrow \text{Stab}(\mathbb{P}^3)$ such that the moduli spaces of semistable objects with Chern character v in its image outside of walls are given in the following order.*

- (i) *The empty space $M_0 = \emptyset$.*
- (ii) *The Grassmannian $M_1 = G(2, 10)$.*
- (iii) *The second moduli space M_2 is the blow up of $G(2, 10)$ in a smooth locus isomorphic to $G(2, 4) \times (\mathbb{P}^3)^\vee$.*
- (iv) *The third moduli space M_3 consists of two irreducible components M_3^1 and M_3^2 . The first component M_3^1 is the blow up of M_2 in the smooth incidence variety parametrizing length-two subschemes in a plane in \mathbb{P}^3 . The second component M_3^2 is a \mathbb{P}^{14} -bundle over $\text{Hilb}^2(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$. The two components intersect transversally in the exceptional locus of the blow up.*
- (v) *The fourth moduli space M_4 has two irreducible components M_4^1 and M_4^2 . The first component is equal to M_3^1 . The second component is birational to M_3^2 .*
- (vi) *The fifth moduli space M_5 is the Hilbert scheme $\text{Hilb}^{4t}(\mathbb{P}^3)$ which has two components H_1 and H_2 . The main component H_1 contains an open subset of elliptic quartic curves and is equal to M_3^1 . The second component is of dimension 23*

and is birational to M_3^2 . Moreover, the two components intersect transversally in a locus of dimension 15.

We believe the use of Bridgeland stability conditions highlights a more general approach that will hopefully lead to more general results about the global geometry of Hilbert schemes of curves in \mathbb{P}^3 . The inevitable occurrence of singularities and more components will make the general situation more involved.

The literature on Hilbert schemes on projective space from a more classical point of view is vast. It turns out that the geometry of these spaces can be quite badly behaved. For example Mumford observed that there is an irreducible component in the Hilbert scheme on \mathbb{P}^3 containing smooth curves that is generically non reduced in [Mum62]. However, Hartshorne proved that Hilbert schemes in projective space are at least connected in [Har66].

In Appendix B we provide code written in [Dev15] to compute walls in tilt stability. This code was used to confirm the computations for both twisted cubics and elliptic quartics

The following theorem connects Bridgeland stability with the simpler notion of tilt stability. It is one of the key ingredients for the theorems above.

Theorem 1.2.4 (See also Theorem 4.1.1). *Let v be the Chern character of an object in $D^b(X)$ such that (v_0, v_1, v_2) is primitive. Then there are two paths $\gamma_1, \gamma_2 : [0, 1] \rightarrow \text{Stab}(\mathbb{P}^3)$ such that all moduli spaces of tilt stable objects with respect to H outside of walls occur as moduli spaces of Bridgeland stable objects along either γ_1 or γ_2 .*

Notice that the theorem does not preclude the existence of further chambers along those paths. In many cases, for example for twisted cubics or elliptic quartics as above,

there are different exact sequences defining identical walls in tilt stability. This happens because the defining objects only differ in the third Chern character. However, by definition changes in ch_3 cannot be detected via tilt stability. In Bridgeland stability those identical walls often move apart and give rise to further chambers.

The computations in tilt stability in this article are very similar in nature to many computations about stability of sheaves on surfaces in [ABCH13, BM14, CHW14, LZ13, MM13, Nue14, Woo13, YY14]. Despite the tremendous success in the surface case, the threefold case has barely been explored.

1.3 Notation

X	smooth projective variety over \mathbb{C} ,
H	fixed ample divisor on X ,
$\mathcal{I}_{Z/X}, \mathcal{I}_Z$	ideal sheaf of a closed subscheme $Z \subset X$,
$D^b(X)$	bounded derived category of coherent sheaves on X ,
$\text{ch}_X(E), \text{ch}(E)$	Chern character of an object $E \in D^b(X)$,
$\text{ch}_{\leq l, X}(E), \text{ch}_{\leq l}(E)$	$(\text{ch}_{0, X}(E), \dots, \text{ch}_{l, X}(E))$,
$H \cdot \text{ch}_X(E), H \cdot \text{ch}(E)$	$(H^n \cdot \text{ch}_{0, X}(E), H^{n-1} \cdot \text{ch}_{1, X}(E), \dots, \text{ch}_{n, X}(E))$ for an ample divisor H on X ,
$H \cdot \text{ch}_{\leq l, X}(E), H \cdot \text{ch}_{\leq l}(E)$	$(H^n \cdot \text{ch}_{0, X}(E), \dots, H^{n-l} \cdot \text{ch}_{l, X}(E))$ for an ample divisor H on X ,
$\text{ch}_X^\beta(E), \text{ch}^\beta(E)$	$e^{-\beta H} \cdot \text{ch}(E)$,
$K_0(X)$	the Grothendieck group of $\text{Coh}(X)$,
Q	smooth quadric hypersurface in \mathbb{P}^4 ,
S	Spinor bundle on Q ,
$\text{ext}^i(E, F)$	$\dim \text{Ext}^i(E, F)$ for $E, F \in D^b(X)$ and $i \in \mathbb{Z}$,
$\text{hom}(E, F)$	$\dim \text{Hom}(E, F)$ for $E, F \in D^b(X)$.

Chapter 2: Preliminaries

In this chapter an introduction to the theory of stability conditions is given. Moreover, some facts on derived categories, Hilbert schemes, moduli spaces of quiver representations and deformation theory are recalled. We will only give proofs if their techniques are useful for the rest of the thesis.

It is assumed that the reader has a working knowledge of modern algebraic geometry as for example laid out in [Har77] (only a fraction of the book will actually be used). We will also assume a basic understanding of Chern characters and the Grothendieck-Riemann-Roch Theorem. For this not more than what is contained in [Har77, Appendix A] is needed, while [Ful98] is the reference on intersection theory that certainly contains more than we will ever need.

2.1 Derived Categories

This section contains definition and important properties of the bounded derived category $D^b(\mathcal{A})$ for an abelian category \mathcal{A} . Most of the time the category \mathcal{A} will be the category $\text{Coh}(X)$ of coherent sheaves on a smooth projective variety X . To simplify notation $D^b(X)$ will be written for $D^b(\text{Coh}(X))$. Derived categories were introduced by Verdier in his thesis [Ver96] in collaboration with his advisor Grothendieck. The

interested reader can find a detailed account of the theory in [GM03], the first two chapters of [Huy06] or the original source [Ver96].

Definition 2.1.1. (i) A complex

$$\dots \rightarrow A^{i-1} \rightarrow A^i \rightarrow A^{i+1} \rightarrow \dots$$

is called *bounded* if $A^i = 0$ for both $i \gg 0$ and $i \ll 0$.

(ii) The objects of the category $\text{Kom}^b(\mathcal{A})$ are bounded complexes over \mathcal{A} and its morphisms are homomorphisms of complexes.

(iii) A morphism $f : A \rightarrow B$ in $\text{Kom}^b(\mathcal{A})$ is called a *quasi-isomorphism* if the induced morphism of cohomology groups $H^i(A) \rightarrow H^i(B)$ is an isomorphism for all integers i .

The bounded derived category of \mathcal{A} is the localization of $\text{Kom}^b(\mathcal{A})$ by quasi-isomorphisms. The exact meaning of this is the next theorem.

Theorem 2.1.2 ([Huy06, Theorem 2.10]). *There is a category $D^b(\mathcal{A})$ together with a functor $Q : \text{Kom}^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ satisfying two properties.*

(i) *The morphism $Q(f)$ is an isomorphism for any quasi-isomorphism f in the category $\text{Kom}^b(\mathcal{A})$.*

(ii) *Any functor $F : \text{Kom}^b(\mathcal{A}) \rightarrow \mathcal{D}$ satisfying property (i) factors uniquely through Q , i.e. there is a unique (up to natural isomorphism) functor $G : D^b(\mathcal{A}) \rightarrow \mathcal{D}$ such that F is naturally isomorphic to $G \circ Q$.*

In particular, Q identifies objects in $\text{Kom}^b(\mathcal{A})$ and $D^b(\mathcal{A})$. By the definition of quasi-isomorphisms we still have well defined cohomology groups $H^i(A)$ for any

$A \in D^b(\mathcal{A})$. The category \mathcal{A} is equivalent to the full subcategory of $D^b(\mathcal{A})$ consisting of those objects $A \in \mathcal{A}$ that satisfy $H^i(A) = 0$ for all $i \neq 0$. In the next section we will learn that this is the simplest example of what is known as the heart of a bounded t-structure.

Notice, there is the automorphism $[1] : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$, where $E[1]$ is defined by $E[1]^i = E^{i+1}$. It simply changes the grading of a complex. Moreover, we define the shift functor $[n] = [1]^n$ for any integer n . The following lemma will be used to actually compute homomorphisms in the derived category.

Lemma 2.1.3 ([Huy06, Proposition 2.56]). *Let \mathcal{A} be either an abelian category with enough injectives or $\text{Coh}(X)$ for a smooth projective variety X . For any $A, B \in \mathcal{A}$ and $i \in \mathbb{Z}$ we have the equality*

$$\text{Hom}_{D^b(\mathcal{A})}(A, B[i]) = \text{Ext}^i(A, B).$$

The category $\text{Coh}(X)$ does usually not have enough injectives, but the category of quasi-coherent sheaves has enough injectives. Therefore, the construction of Ext-groups and the above lemma is more involved for $\mathcal{A} = \text{Coh}(X)$. In contrast to $\text{Kom}^b(\mathcal{A})$ the bounded derived category $D^b(\mathcal{A})$ is not abelian. This led Verdier and Grothendieck to the notion of a triangulated category, which will be explained in the next theorem.

Definition 2.1.4. For any morphism $f : A \rightarrow B$ in $\text{Kom}^b(\mathcal{A})$ the cone $C(f)$ is defined by $C(f)^i = A^{i+1} \oplus B^i$. The differential is given by the matrix

$$\begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}.$$

The inclusion $B^i \hookrightarrow A^{i+1} \oplus B^i$ leads to a morphism $B \rightarrow C(f)$ and the projection $B^i \oplus A^{i+1} \twoheadrightarrow A^{i+1}$ leads to a morphism $C(f) \rightarrow A[1]$.

Definition 2.1.5. A sequence of maps $F \rightarrow E \rightarrow G \rightarrow F[1]$ in $D^b(\mathcal{A})$ is called a *distinguished triangle* if there is a morphism $f : A \rightarrow B$ in $\text{Kom}^b(\mathcal{A})$ and a commutative diagram with vertical isomorphisms in $D^b(\mathcal{A})$ as follows

$$\begin{array}{ccccccc} F & \longrightarrow & E & \longrightarrow & G & \longrightarrow & F[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \longrightarrow & C(f) & \longrightarrow & A[1]. \end{array}$$

These distinguished triangles should be viewed as the analogue of exact sequences in an abelian category. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence in \mathcal{A} , then $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle where the map $C \rightarrow A[1]$ is determined by the element in $\text{Hom}(C, A[1]) = \text{Ext}^1(C, A)$ that determines the extension B . The following properties of the derived category are essentially the defining properties of a *triangulated category*.

Theorem 2.1.6 ([GM03, Chapter IV]). (i) Any morphism $A \rightarrow B$ in $D^b(\mathcal{A})$ can be completed to a distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$.

(ii) A triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ is distinguished if and only if the induced triangle $B \rightarrow C \rightarrow A[1] \rightarrow B[1]$ is distinguished.

(iii) Assume we have two distinguished triangles with morphisms f and g making the diagram below commutative.

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow \exists h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1]. \end{array}$$

Then we can find $h : C \rightarrow C'$ making the whole diagram commutative.

(iv) Assume we have two morphisms $A \rightarrow B$ and $B \rightarrow C$. Then together with (i) and (iii) we can get a commutative diagram as follows where all rows and columns are distinguished triangles.

$$\begin{array}{ccccccc}
 A & \longrightarrow & B & \longrightarrow & D & \longrightarrow & A[1] \\
 \downarrow \text{id} & & \downarrow & & \downarrow \exists & & \downarrow \text{id}[1] \\
 A & \longrightarrow & C & \longrightarrow & E & \longrightarrow & A[1] \\
 \downarrow & & \downarrow & & \downarrow \exists & & \downarrow \\
 0 & \longrightarrow & F & \xrightarrow{\text{id}} & F & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \exists & & \downarrow \\
 A[1] & \longrightarrow & B[1] & \longrightarrow & D[1] & \longrightarrow & A[2]
 \end{array}$$

The key in property (iv) is that the triangle $D \rightarrow E \rightarrow F \rightarrow D[1]$ is actually distinguished. Be aware that contrary to most definitions in category theory the morphism in (iii) is not necessarily unique.

All triangles coming up in the rest of this thesis are distinguished. Therefore, we will simply drop the word distinguished from the notation.

2.2 Stability Conditions

In this section various definitions of stability in the derived category are explained. The most general notion is that of a very weak stability condition, introduced in Appendix B of [BMS14]. We will describe this notion more closely to how it was defined in [PT15]. The main examples are those of classical slope stability and tilt stability. The latter was proposed as a stepping stone to the definition of Bridgeland stability on threefolds in [BMT14]. We will recall this conjectural construction. The content of this section is similar to the preliminary section in [Sch15].

2.2.1 Very Weak Stability Conditions and the Support Property

The notion of a very weak stability condition encompasses all forms of stability used in this thesis. It will provide the possibility to treat different forms of stability uniformly. At the center of the theory are abelian subcategories of $D^b(X)$ known as hearts of bounded t-structures, where X is a smooth projective variety. The idea due to Bridgeland is to vary the category $\text{Coh}(X)$ instead of just the slope function as done in classical slope stability. We recommend the reader to keep the example of classical slope stability in mind when reading this chapter. We will embed it in full details into this theory at the very end of this subsection.

Definition 2.2.1. The *heart of a bounded t-structure* on $D^b(X)$ is a full additive subcategory $\mathcal{A} \subset D^b(X)$ such that

- for integers $i > j$ and $A \in \mathcal{A}[i]$, $B \in \mathcal{A}[j]$ the vanishing $\text{Hom}(A, B) = 0$ holds,
- for all $E \in D^b(X)$ there are integers $k_1 > \dots > k_m$, objects $E_i \in D^b(X)$, $A_i \in \mathcal{A}$ for $i = 1, \dots, m$ and a collection of triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots & \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & A_1[k_1] & & A_2[k_2] & & & & A_{m-1}[k_{m-1}] & & A_m[k_m].
 \end{array}$$

The heart of a bounded t-structure is automatically abelian. A proof of this fact and a full introduction to the theory of t-structures can be found in [BBD82]. The standard example of a heart of a bounded t-structure on $D^b(X)$ is given by $\text{Coh}(X)$. If $D^b(\mathcal{A}) \cong D^b(X)$, then \mathcal{A} is the heart of a bounded t-structure. The converse does not hold, but this is still one of the most important examples for having an intuition about this notion. A slicing further refines the heart of a bounded t-structure.

Definition 2.2.2 ([Bri07]). A *slicing* of $D^b(X)$ is a collection of subcategories $P(\phi) \subset D^b(X)$ for all $\phi \in \mathbb{R}$ such that

- $P(\phi)[1] = P(\phi + 1)$,
- if $\phi_1 > \phi_2$ and $A \in P(\phi_1)$, $B \in P(\phi_2)$ then $\text{Hom}(A, B) = 0$,
- for all $E \in D^b(X)$ there are real numbers $\phi_1 > \dots > \phi_m$, objects $E_i \in D^b(X)$, $A_i \in P(\phi_i)$ for $i = 1, \dots, m$ and a collection of triangles

$$\begin{array}{ccccccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots & \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & A_1 & & A_2 & & & & A_{m-1} & & A_m
 \end{array}$$

where $A_i \in P(\phi_i)$.

For this filtration of an element $E \in D^b(X)$ we write $\phi^-(E) := \phi_m$ and $\phi^+(E) := \phi_1$ (see below for well definedness). Moreover, for $E \in P(\phi)$ we call $\phi(E) := \phi$ the *phase* of E .

The last property is called the *Harder-Narasimhan filtration*. By setting $\mathcal{A} := P((0, 1])$ to be the extension closure of the subcategories $\{P(\phi) : \phi \in (0, 1]\}$ one gets the heart of a bounded t-structure from a slicing. In both cases of a slicing and the heart of a bounded t-structure the vanishing of morphisms allows one to show that the Harder-Narasimhan filtration is unique via induction on its length.

Let $v : K_0(X) \rightarrow \Gamma$ be a fixed homomorphism where Γ is a finite rank \mathbb{Z} -lattice. Note that any complex $A^\bullet \in D^b(X)$ maps to the element $\sum_i (-1)^i [A^i] \in K_0(X)$. Fix H to be an ample divisor on X . Then v will usually be one of the homomorphisms $H \cdot \text{ch}_{\leq l}$ defined by

$$E \mapsto (H^n \cdot \text{ch}_0(E), \dots, H^{n-l} \cdot \text{ch}_l(E)).$$

for some $l \leq n$.

Definition 2.2.3 ([BMS14, PT15]). A *very weak pre-stability condition* on $D^b(X)$ is a pair $\sigma = (P, Z)$ where P is a slicing of $D^b(X)$ and $Z : \Gamma \rightarrow \mathbb{C}$ is a homomorphism usually called the *central charge* such that any non zero $E \in P(\phi)$ satisfies

$$Z(v(E)) \in \begin{cases} \mathbb{R}_{>0}e^{i\pi\phi} & \text{for } \phi \in \mathbb{R} \setminus \mathbb{Z} \\ \mathbb{R}_{\geq 0}e^{i\pi\phi} & \text{for } \phi \in \mathbb{Z}. \end{cases}$$

This definition is short and good for abstract argumentation, but it is not very practical for defining concrete examples. As before, the heart of a bounded t-structure can be defined by $\mathcal{A} := P((0, 1])$. Equivalently, it is possible to define a very weak pre-stability condition via the heart of a bounded t-structure \mathcal{A} and a central charge $Z : \Gamma \rightarrow \mathbb{C}$ such that $Z \circ v$ maps $\mathcal{A} \setminus \{0\}$ to the upper half plane plus the non positive real line $\{re^{i\pi\varphi} : r \geq 0, \varphi \in (0, 1]\}$. Moreover, one needs to demand that Harder-Narasimhan filtrations exist inside \mathcal{A} to actually get a very weak pre-stability condition. This makes sense, because we can define a slope function by

$$\mu_\sigma := -\frac{\Re(Z)}{\Im(Z)},$$

where dividing by 0 is interpreted as $+\infty$. Then an object $E \in \mathcal{A}$ is called *(semi)stable* if for all monomorphisms $A \hookrightarrow E$ in \mathcal{A} we have $\mu_\sigma(A) < (\leq) \mu_\sigma(A/E)$. More generally, an element $E \in D^b(X)$ is called *(semi)stable* if there is $m \in \mathbb{Z}$ such that $E[m] \in \mathcal{A}$ is *(semi)stable*. A semistable but not stable object is called *strictly semistable*.

To go back from this definition to a slicing, we define the subcategories $P(\phi)$ for $\phi \in (0, 1]$ to consist of all semistable objects $E \in \mathcal{A}$ such that

$$Z(v(E)) \in \begin{cases} \mathbb{R}_{>0}e^{i\pi\phi} & \text{for } \phi \in \mathbb{R} \setminus \mathbb{Z} \\ \mathbb{R}_{\geq 0}e^{i\pi\phi} & \text{for } \phi \in \mathbb{Z}. \end{cases}$$

In the following we interchangeably use (\mathcal{A}, Z) and (P, Z) to denote the same very weak pre-stability condition. An important tool is the support property. It was introduced in [KS08] for Bridgeland stability conditions, but can be adapted without much trouble to very weak stability conditions (see [PT15, Section 2]). We also recommend [BMS14, Appendix A] for a nicely written treatment of this notion. Without loss of generality, we can assume that $Z(v(E)) = 0$ implies $v(E) = 0$. If not, we replace Γ by a suitable quotient.

Definition 2.2.4. A very weak pre-stability condition $\sigma = (\mathcal{A}, Z)$ satisfies the *support property* if there is a symmetric bilinear form Q on $\Gamma \otimes \mathbb{R}$ such that

- (i) all semistable objects $E \in \mathcal{A}$ satisfy the inequality $Q(v(E), v(E)) \geq 0$ and
- (ii) all non zero vectors $v \in \Gamma \otimes \mathbb{R}$ with $Z(v) = 0$ satisfy $Q(v, v) < 0$.

A very weak pre-stability condition satisfying the support property is called a *very weak stability condition*.

The inequality $Q(v(E), v(E)) \geq 0$ can be viewed as some generalization of the classical Bogomolov inequality for vector bundles. By abuse of notation we will write $Q(E, F)$ instead of $Q(v(E), v(F))$ for $E, F \in \mathcal{A}$. We will also use the notation $Q(E) = Q(E, E)$.

Let $\text{Stab}^{vw}(X, v)$ be the set of very weak stability conditions on X with respect to v . This set can be given a topology as the coarsest topology such that for any $E \in D^b(X)$ the maps $(\mathcal{A}, Z) \mapsto Z$, $(\mathcal{A}, Z) \mapsto \phi^+(E)$ and $(\mathcal{A}, Z) \mapsto \phi^-(E)$ are continuous.

Lemma 2.2.5 ([BMS14][Lemma 3.9, Section 8, Lemma A.7 & Proposition A.8]).

Assume that Q has signature $(2, \text{rk } \Gamma - 2)$ and U is a path-connected open subset of $\text{Stab}^{vw}(X, v)$ such that all $\sigma \in U$ satisfy the support property with respect to Q .

- If $E \in D^b(X)$ with $Q(E) = 0$ is σ -stable for some $\sigma \in U$ then it is σ' -stable for all $\sigma' \in U$ unless it is destabilized by an object F with $v(F) = 0$.
- Let ρ be a ray in \mathbb{C} starting at the origin. Then

$$\mathcal{C}^+ = Z^{-1}(\rho) \cap \{Q \geq 0\}$$

is a convex cone for any very weak stability condition $(\mathcal{A}, Z) \in U$.

- If $v, v_1, v_2 \in \mathcal{C}^+$ with $v = v_1 + v_2$ and $Q(v) = Q(v_i)$ for $i = 1$ or $i = 2$, then $Q(v) = Q(v_1) = Q(v_2) = 0$.
- Any vector $w \in \mathcal{C}^+$ with $Q(w) = 0$ generates an extremal ray of \mathcal{C}^+ .

Only the situation of an actual stability condition (we will define this in Subsection 2.2.3) is handled in [BMS14]. In that situation there are no objects F in the heart with $v(F) = 0$. However, exactly the same arguments go through in the case of a very weak stability condition.

Definition 2.2.6. A *numerical wall* inside $\text{Stab}^{vw}(X, v)$ (or a subspace of it) with respect to an element $w \in \Gamma$ is a proper non trivial solution set of those σ satisfying the equation $\mu_\sigma(w) = \mu_\sigma(u)$ for a fixed element $u \in \Gamma$.

A subset of a numerical wall is called an *actual wall* if for each point of the subset there is an exact sequence of semistable objects $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ in \mathcal{A} where $v(E) = w$ and $\mu_\sigma(F) = \mu_\sigma(G)$ defines the numerical wall.

After this chapter we will only deal with actual walls and simply call them walls. Both actual and numerical walls in the space of very weak stability conditions satisfy certain numerical restrictions with respect to Q .

Lemma 2.2.7. *Let $\sigma = (\mathcal{A}, Z)$ be a very weak stability condition satisfying the support property with respect to Q (it is actually enough for Q to be negative semi-definite on $\text{Ker } Z$).*

(i) *Let $F, G \in \mathcal{A}$ be semistable objects. If $\mu(F) = \mu(G)$, then $Q(F, G) \geq 0$.*

(ii) *Assume there is an actual wall defined by an exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$. Then $0 \leq Q(F) + Q(G) \leq Q(E)$.*

Proof. We start with the first statement. If $Z(F) = 0$ or $Z(G) = 0$, then $Q(F, G) = 0$. If not, there is $\lambda > 0$ such that $Z(F - \lambda G) = 0$. Therefore, we get

$$0 \geq Q(F - \lambda G) = Q(F) + \lambda^2 Q(G) - 2\lambda Q(F, G).$$

The inequalities $Q(F) \geq 0$ and $Q(G) \geq 0$ lead to $Q(F, G) \geq 0$. For the second statement we have

$$Q(E) = Q(F) + Q(G) + 2Q(F, G) \geq 0.$$

Since all four terms are positive, the claim follows. □

Remark 2.2.8. Since Q only has to be negative semi-definite on $\text{Ker } Z$ for the Lemma to apply, it is sometimes possible to define Q on a bigger lattice than Γ . For example, we will define a very weak stability condition factoring through $v = H \cdot \text{ch}_{\leq 2}$, but apply the Lemma for $v = H \cdot \text{ch}$ where everything is still well defined later on.

The best known example of a very weak stability condition is slope stability. We will slightly generalize it for notational purposes. Let H be a fixed ample divisor on X . Moreover, pick a real number β . Then the twisted Chern character ch^β is defined to be $e^{-\beta H} \cdot \text{ch}$. We will later on only deal with the case $\dim X = 3$ where one has

$$\begin{aligned}\text{ch}_0^\beta &= \text{ch}_0, \\ \text{ch}_1^\beta &= \text{ch}_1 - \beta H \cdot \text{ch}_0, \\ \text{ch}_2^\beta &= \text{ch}_2 - \beta H \cdot \text{ch}_1 + \frac{\beta^2}{2} H^2 \cdot \text{ch}_0, \\ \text{ch}_3^\beta &= \text{ch}_3 - \beta H \cdot \text{ch}_2 + \frac{\beta^2}{2} H^2 \cdot \text{ch}_1 - \frac{\beta^3}{6} H^3 \cdot \text{ch}_0.\end{aligned}$$

In this case $v = H \cdot \text{ch}_{\leq 1} : K_0(X) \rightarrow \mathbb{Z}^2$. The central charge $Z_\beta^{sl} : \mathbb{Z}^2 \rightarrow \mathbb{C}$ is given by

$$Z_\beta^{sl}(r, c) = -(c - \beta r) + ir.$$

The heart of a bounded t-structure in this case is simply $\text{Coh}(X)$. The existence of Harder-Narasimhan filtration was first proven for curves in [HN75], but holds in general. Finally the support property is satisfied for $Q = 0$. We will denote the corresponding slope function by

$$\mu_\beta := \frac{H^{n-1} \cdot \text{ch}_1^\beta}{H^n \cdot \text{ch}_0^\beta} = \frac{H^{n-1} \cdot \text{ch}_1}{H^n \cdot \text{ch}_0} - \beta,$$

where $n = \dim X$. Note that the modification by β does not change stability itself but just shifts the value of the slope.

2.2.2 Tilt Stability

In [BMT14] the notion of tilt stability has been introduced as an auxiliary notion in between classical slope stability and Bridgeland stability on threefolds. A precise

definition of what a Bridgeland stability condition is will occur in the next subsection. We will recall the construction of tilt stability and prove a few properties. From now on let $\dim X = 3$.

The process of tilting is used to obtain a new heart of a bounded t-structure. For more information on the general theory of tilting we refer to [HRS96]. A torsion pair is defined by

$$\mathcal{T}_\beta = \{E \in \text{Coh}(X) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_\beta(G) > 0\},$$

$$\mathcal{F}_\beta = \{E \in \text{Coh}(X) : \text{any subsheaf } F \subset E \text{ satisfies } \mu_\beta(F) \leq 0\}.$$

A new heart of a bounded t-structure is defined as the extension closure $\text{Coh}^\beta(X) := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$. In this case $v = H \cdot \text{ch}_{\leq 2} : K_0(X) \rightarrow \mathbb{Z}^2 \oplus \frac{1}{2}\mathbb{Z}$. Let $\alpha > 0$ be a positive real number. The central charge is given by

$$Z_{\alpha,\beta}^{\text{tilt}}(r, c, d) = -(d - \beta c + \frac{\beta^2}{2}r) + \frac{\alpha^2}{2}r + i(c - \beta r)$$

The corresponding slope function is

$$\nu_{\alpha,\beta} := \frac{H \cdot \text{ch}_2^\beta - \frac{\alpha^2}{2}H^3 \cdot \text{ch}_0^\beta}{H^2 \cdot \text{ch}_1^\beta}.$$

Note that in regard to [BMT14] this slope has been modified by reparametrizing ω as $\sqrt{3}\omega$. We prefer this point of view for aesthetical reasons, because it will make the numerical walls semicircles and not just ellipses. Every object in $\text{Coh}^\beta(X)$ has a Harder-Narasimhan filtration due to [BMT14, Lemma 3.2.4]. The support property is directly linked to the Bogomolov inequality. This inequality was first proven for slope semistable sheaves in [Bog78]. We define the bilinear form for the support property by $Q^{\text{tilt}}((r, c, d), (R, C, D)) = Cc - Rd - Dr$.

Theorem 2.2.9 (Bogomolov Inequality for Tilt Stability, [BMT14, Corollary 7.3.2]).

Any $\nu_{\alpha,\beta}$ -semistable object $E \in \text{Coh}^\beta(X)$ satisfies

$$\begin{aligned} Q^{\text{tilt}}(E) &= (H^2 \cdot \text{ch}_1^\beta(E))^2 - 2(H^3 \cdot \text{ch}_0^\beta)(H \cdot \text{ch}_2^\beta) \\ &= (H^2 \cdot \text{ch}_1(E))^2 - 2(H^3 \cdot \text{ch}_0)(H \cdot \text{ch}_2) \geq 0. \end{aligned}$$

As a consequence $(\text{Coh}^\beta, Z_{\alpha,\beta}^{\text{tilt}})$ satisfies the support property with respect to Q^{tilt} . On smooth projective surfaces this is already enough to get a Bridgeland stability condition (see [Bri08, AB13]). On threefolds this notion is not able to properly handle geometry that occurs in codimension three as we will see.

Proposition 2.2.10 ([BMS14, Appendix B]). *The function $\mathbb{R}_{>0} \times \mathbb{R} \rightarrow \text{Stab}^{vw}(X, v)$ defined by $(\alpha, \beta) \mapsto (\text{Coh}^\beta(X), Z_{\alpha,\beta}^{\text{tilt}})$ is continuous. Moreover, actual walls with respect to a class $w \in \Gamma$ in the image of this map are locally finite, i.e. any (α, β) has an open neighborhood with only finitely many actual walls.*

Numerical walls in tilt stability satisfy Bertram's Nested Wall Theorem. For surfaces it was the main result in [Mac14a].

Theorem 2.2.11 (Structure Theorem for Walls in Tilt Stability). *Let $(R, C, D) \in \mathbb{Z}^2 \times \frac{1}{2}\mathbb{Z}$ be a fixed vector. All numerical walls in the following statements are with respect to (R, C, D) .*

(i) *Numerical walls in tilt stability are of the form*

$$x\alpha^2 + x\beta^2 + y\beta + z = 0$$

for $x = Rc - Cr$, $y = 2(Dr - Rd)$ and $z = 2(Cd - Dc)$. In particular, they are either semicircles with center on the β -axis or vertical rays.

(ii) If two numerical walls given by the equations $\nu_{\alpha,\beta}(r, c, d) = \nu_{\alpha,\beta}(R, C, D)$ and $\nu_{\alpha,\beta}(r', c', d') = \nu_{\alpha,\beta}(R, C, D)$ intersect for any $\alpha \geq 0$ and $\beta \in \mathbb{R}$ then (r, c, d) , (r', c', d') and (R, C, D) are linearly dependent. In particular, the two numerical walls are identical.

(iii) The curve $\nu_{\alpha,\beta}(R, C, D) = 0$ is given by the hyperbola

$$R\alpha^2 - R\beta^2 + 2C\beta - 2D = 0.$$

Moreover, this hyperbola intersects all semicircles at their top point.

(iv) If $R \neq 0$, there is exactly one vertical numerical wall given by $\beta = C/R$. If $R = 0$, there is no vertical actual wall.

(v) If a numerical wall has a single point at which it is an actual wall, then all of it is an actual wall.

Proof. Part (i) and (iii) are straightforward but lengthy computations only relying on the numerical data.

A numerical wall can also be described as two vectors mapping to the same line under the \mathbb{R} -linear map $Z_{\alpha,\beta}^{\text{tilt}}$. This function maps surjectively onto \mathbb{C} . Therefore, at most two linearly independent vectors can be mapped onto the same line. That proves (ii).

In order to prove (iv), observe that a vertical numerical wall occurs when $x = 0$ holds. By the above formula for x this implies

$$c = \frac{Cr}{R}$$

in case $R \neq 0$. A direct computation shows that the equation simplifies to $\beta = C/R$. If $R = 0$ and $C \neq 0$, then $r = 0$. This implies that the two slopes are the same

for all or no (α, β) . If $R = C = 0$, then all objects with this Chern character are automatically semistable and there are no actual walls at all.

Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be an exact sequence of tilt semistable objects in $\text{Coh}^\beta(X)$ that defines a numerical wall. If there is a point on the numerical wall at which this sequence does not define an actual wall anymore, then either F , E or G have to destabilize at another point along the numerical wall in between the two points. But that would mean two numerical walls intersect in contradiction to (ii). \square

A generalized Bogomolov type inequality involving third Chern characters for tilt semistable objects with $\nu_{\alpha, \beta} = 0$ has been conjectured in [BMT14]. In [BMS14] it was shown that the conjecture is equivalent to the following more general inequality that drops the hypothesis $\nu_{\alpha, \beta} = 0$.

Conjecture 2.2.12 (BMT Inequality). *Any $\nu_{\alpha, \beta}$ -semistable object $E \in \text{Coh}^\beta(X)$ satisfies*

$$\alpha^2 Q^{\text{tilt}}(E) + 4(H \cdot \text{ch}_2^\beta(E))^2 - 6(H^2 \cdot \text{ch}_1^\beta(E)) \text{ch}_3^\beta(E) \geq 0.$$

By using the definition of $\text{ch}^\beta(E)$ and expanding the expression one can find $x(E), y(E) \in \mathbb{R}$ depending on E such that the inequality becomes

$$\alpha^2 Q^{\text{tilt}}(E) + \beta^2 Q^{\text{tilt}}(E) + x(E)\beta + y(E) \geq 0.$$

This means the solution set is given by the complement of a semi-disc with center on the β -axis or a quadrant to one side of a vertical line. The conjecture is known for \mathbb{P}^3 [Mac14b], the smooth quadric threefold [Sch14] and all abelian threefolds [BMS14, MP15, MP16]. Recently, the former two results were generalized to all Fano threefolds

of Picard rank 1 in [Li15]. In the next section, we will provide a counterexample in the case X is the blow up of \mathbb{P}^3 in one point.

Another question that comes up in concrete situations is whether a given tilt semistable object is a sheaf. For a fixed β let

$$c := \inf\{H^2 \cdot \text{ch}_1^\beta(E) > 0 : E \in \text{Coh}^\beta(X)\}.$$

Lemma 2.2.13 ([BMT14, Lemma 7.2.1 and 7.2.2]). *An object $E \in \text{Coh}^\beta(X)$ that is $\nu_{\alpha,\beta}$ -semistable for all $\alpha \gg 0$ is given by one of three possibilities.*

- (i) $E = H^0(E)$ is a pure sheaf supported in dimension greater than or equal to two that is slope semistable.
- (ii) $E = H^0(E)$ is a sheaf supported in dimension less than or equal to one.
- (iii) $H^{-1}(E)$ is a torsion free slope semistable sheaf and $H^0(E)$ is supported in dimension less than or equal to one. Moreover, if $\mu_\beta(E) < 0$ then $\text{Hom}(F, E) = 0$ for all sheaves F of dimension less than or equal to one.

An object $F \in \text{Coh}^\beta(X)$ with $H^2 \cdot \text{ch}_1^\beta \in \{0, c\}$ is $\nu_{\alpha,\beta}$ -semistable if and only if it is given by one of the three types above.

Notice that part of the second statement follows directly from the first as follows. Any subobject of F in $\text{Coh}^\beta(X)$ must have $H^2 \cdot \text{ch}_1^\beta = 0$ or $H^2 \cdot \text{ch}_1^\beta = c$. In the second case the corresponding quotient satisfies $H^2 \cdot \text{ch}_1^\beta = 0$. Therefore, in both cases either the quotient or the subobject have infinite slope. This means there is no actual wall that could destabilize F for any $\alpha > 0$. This type of argument will be used several times in the next sections. Using the same proof as in the surface case in [Bri08, Proposition 14.1] leads to the following lemma.

Lemma 2.2.14. *Assume $E \in \text{Coh}(X)$ is a slope stable sheaf and $\beta < \mu(E)$. Then E is $\nu_{\alpha,\beta}$ -stable for all $\alpha \gg 0$.*

The functor $\mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_X)[1]$ is a contravariant autoequivalence of the derived category $D^b(X)$. The following statement describes how tilt stability behaves under this autoequivalence.

Proposition 2.2.15 ([BMT14, Proposition 5.1.3]). *Assume $E \in \text{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$ -semistable with $\nu_{\alpha,\beta}(E) \neq \infty$. Then there is a $\nu_{\alpha,-\beta}$ -semistable object $\tilde{E} \in \text{Coh}^{-\beta}(X)$ and a sheaf T supported in dimension 0 together with a triangle*

$$\tilde{E} \rightarrow \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)[1] \rightarrow T[-1] \rightarrow \tilde{E}[1].$$

2.2.3 Bridgeland Stability

We will recall the definition of a Bridgeland stability condition from [Bri07] and show how they can be conjecturally constructed on threefolds based on the BMT-inequality as described in [BMT14]. It is known that the inequality holds on $X = \mathbb{P}^3$ due to [Mac14b] and we will apply it in a later section to study concrete examples of moduli spaces of complexes in this case. Moreover, we will give a proof of the conjecture in the case of a smooth quadric hypersurface in \mathbb{P}^4 in the next chapter.

Definition 2.2.16. A *Bridgeland (pre-)stability condition* on the category $D^b(X)$ is a very weak (pre-)stability condition (P, Z) such that $Z(E) \neq 0$ for all semistable objects $E \in D^b(X)$. By $\text{Stab}(X, v)$ we denote the subspace of Bridgeland stability conditions in $\text{Stab}^{vw}(X, v)$.

If $\mathcal{A} = P((0, 1])$ is the corresponding heart, then we could have equivalently defined a Bridgeland stability condition by the property $Z(E) \neq 0$ for all non zero

$E \in \mathcal{A}$. Instead of setting $\mathcal{A} = P((0, 1])$, we could have also used the heart $\mathcal{A} = P((\phi - 1, \phi])$ for any $\phi \in \mathbb{R}$ without substantial differences. In some very special cases it is possible to choose ϕ such that the corresponding heart is equivalent to the category of representations of a quiver with relations. This will be particularly useful in the case of \mathbb{P}^3 .

Theorem 2.2.17 ([Bri07, Section 7]). *The map $(\mathcal{A}, Z) \mapsto Z$ from $\text{Stab}(X, v)$ to $\text{Hom}(\Gamma, \mathbb{C})$ is a local homeomorphism. In particular, $\text{Stab}(X, v)$ is a complex manifold.*

In order to have any hope of actually computing wall-crossing behaviour it is necessary for walls in Bridgeland stability to be somewhat reasonably behaved. The following result due to [Bri08, Section 9] is a major step towards that.

Theorem 2.2.18. *The set of actual walls in Bridgeland stability is locally finite, i.e. for a fixed vector $w \in \Gamma$ there are only finitely many actual walls in any compact subset of $\text{Stab}(X, v)$. In particular, being stable is an open property in $\text{Stab}(X, v)$, i.e. if an object $E \in D^b(X)$ is stable at a point in $\text{Stab}(X, v)$ it is stable in an open neighborhood of that point.*

An important question is how moduli spaces change set theoretically at actual walls. In case the destabilizing subobject and quotient are both stable this has a satisfactory answer due to [BM11, Lemma 5.9]. This proof does not work in the case of very weak stability conditions due to the lack of unique factors in the Jordan-Hölder filtration.

Lemma 2.2.19. *Let $\sigma = (\mathcal{A}, Z) \in \text{Stab}(X)$ such that there are stable object $F, G \in \mathcal{A}$ with $\mu_\sigma(F) = \mu_\sigma(G)$. Then there is an open neighborhood U around σ where*

non trivial extensions $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ are stable for all $\sigma' \in U$ such that $\phi_{\sigma'}(F) < \phi_{\sigma'}(G)$.

Proof. Since stability is an open property there is an open neighborhood U of σ in which both F and G are stable. The category $P(\phi_{\sigma}(F))$ is of finite length with simple objects corresponding to stable objects. In fact $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is a Jordan-Hölder filtration. By shrinking U if necessary we know that if E is unstable at a point in U , there is a sequence $0 \rightarrow F' \rightarrow E \rightarrow G' \rightarrow 0$ that becomes a Jordan-Hölder filtration at σ . Since the Jordan-Hölder filtration has unique factors up to order and E is a non trivial extension, we get $F = F'$ and $G = G'$. Therefore, there is no destabilizing sequence if $\phi_{\sigma'}(F) < \phi_{\sigma'}(G)$. \square

It turns out that while constructing very weak stability conditions is not very difficult, constructing Bridgeland stability conditions is in general a wide open problem. For any smooth projective variety of dimension bigger than or equal to two, there is no Bridgeland stability condition factoring through the Chern character for $\mathcal{A} = \text{Coh}(X)$ due to [Tod09, Lemma 2.7].

Tilt stability is not Bridgeland stability as can be seen by the fact that skyscraper sheaves are mapped to the origin. In [BMT14] it was conjectured that one has to tilt $\text{Coh}^{\beta}(X)$ again as follows in order to construct a Bridgeland stability condition on a threefold. Let

$$\mathcal{T}_{\alpha,\beta} = \{E \in \text{Coh}^{\beta}(X) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\},$$

$$\mathcal{F}_{\alpha,\beta} = \{E \in \text{Coh}^{\beta}(X) : \text{any subobject } F \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta}(F) \leq 0\}$$

and set $\mathcal{A}^{\alpha,\beta}(X) := \langle \mathcal{F}_{\alpha,\beta}[1], \mathcal{T}_{\alpha,\beta} \rangle$. For any $s > 0$ they define

$$\begin{aligned} Z_{\alpha,\beta,s} &:= -\text{ch}_3^\beta + (s + \frac{1}{6})\alpha^2 H^2 \cdot \text{ch}_1^\beta + i(H \cdot \text{ch}_2^\beta - \frac{\alpha^2}{2} H^3 \cdot \text{ch}_0^\beta), \\ \lambda_{\alpha,\beta,s} &:= -\frac{\Re(Z_{\alpha,\beta,s})}{\Im(Z_{\alpha,\beta,s})}. \end{aligned}$$

In this case the bilinear form is given by

$$\begin{aligned} Q_{\alpha,\beta,K}((r, c, d, e), (R, C, D, E)) &:= Q^{\text{tilt}}((r, c, d), (R, C, D))(K\alpha^2 + \beta^2) \\ &\quad + (3Er + 3Re - Cd - Dc)\beta \\ &\quad - 3Ce - 3Ec + 4Dd. \end{aligned}$$

for some $K \in (1, 6s+1)$. Notice that for $K = 1$ this comes directly from the quadratic form in the BMT-inequality.

Theorem 2.2.20 ([BMT14, Corollary 5.2.4], [BMS14, Lemma 8.8]). *Let $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$. The BMT-inequality holds for all tilt semistable $E \in \text{Coh}^\beta(X)$ with $\nu_{\alpha,\beta}(E) = 0$ if and only if $(\mathcal{A}^{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ is a Bridgeland pre-stability condition for all $s > 0$.*

If the BMT-inequality holds for all tilt semistable $E \in \text{Coh}^\beta(X)$, then the support property is satisfied with respect to $Q_{\alpha,\beta,K}$ for any $K \in (1, 6s+1)$.

The proof of this theorem is based on the fact that the BMT-inequality for objects of tilt slope zero translates to the property that $Z_{\alpha,\beta,s}$ does not map to 0 or the positive real line. Note that as a consequence the BMT inequality holds for all $\lambda_{\alpha,\beta,s}$ -stable objects. In [BMS14, Proposition 8.10] it is shown that this implies a continuity result just as in the case of tilt stability.

Proposition 2.2.21. *The function $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0} \rightarrow \text{Stab}(X, v)$ defined by $(\alpha, \beta, s) \mapsto (\mathcal{A}^{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ is continuous.*

In the case of tilt stability we have seen that the limiting stability for $\alpha \rightarrow \infty$ is closely related with slope stability. The first step in connecting Bridgeland stability with tilt stability is a similar result. If $F \in \text{Coh}^\beta(X)[i]$ is a factor in the Harder-Narasimhan filtration of an object $E \in D^b(X)$ with respect to the heart $\text{Coh}^\beta(X)$, then we define the i -th cohomology of E with respect to $\text{Coh}^\beta(X)$ by $\mathcal{H}_\beta^i(E) = F[-i]$.

Lemma 2.2.22 ([BMS14, Lemma 8.9]). *If $E \in \mathcal{A}^{\alpha,\beta}(X)$ is $Z_{\alpha,\beta,s}$ -semistable for all $s \gg 0$, then one of the following two conditions holds.*

(i) $E = \mathcal{H}_\beta^0(E)$ is a $\nu_{\alpha,\beta}$ -semistable object.

(ii) $\mathcal{H}_\beta^{-1}(E)$ is $\nu_{\alpha,\beta}$ -semistable and $\mathcal{H}_\beta^0(E)$ is a sheaf supported in dimension 0.

2.3 Moduli Spaces

In this section we will define and describe properties of various moduli spaces. They are the main reason to study any type of stability conditions. Roughly, a moduli space is a space whose points parametrize some kind of objects. There is no precise definition, but you know one if you see one. We will in fact give a definition, but it is much broader than what one intuitively views as a moduli space.

For convenience we restrict ourselves to schemes over the complex numbers even though some of the followings results hold in larger generality. Let $\underline{\text{Sch}}$ be the category of finite type schemes over $\text{Spec}(\mathbb{C})$. Moreover, $\underline{\text{Set}}$ is the category of sets.

Definition 2.3.1. Let $\mathcal{F} : \underline{\text{Sch}} \rightarrow \underline{\text{Set}}$ be a contravariant functor.

- (i) A scheme $X \in \underline{\text{Sch}}$ represents \mathcal{F} or is the *fine moduli space* with respect to \mathcal{F} , if there is a natural isomorphism $\mathcal{F} \cong \text{Hom}(\cdot, X)$.

- (ii) If X is a fine moduli space with respect to \mathcal{F} , the object $\mathcal{U} \in \mathcal{F}(X)$ corresponding to the identity in $\text{Hom}(X, X)$ is called the *universal family*.
- (iii) A scheme $X \in \underline{\text{Sch}}$ is the *coarse moduli space* with respect to \mathcal{F} , if there is a natural transformation $\mathcal{F} \rightarrow \text{Hom}(\cdot, X)$ such that any other natural transformation $\mathcal{F} \rightarrow \text{Hom}(\cdot, X')$ for another $X' \in \underline{\text{Sch}}$ factors through $\text{Hom}(\cdot, X) \rightarrow \text{Hom}(\cdot, X')$ induced via a morphism $X \rightarrow X'$.

Both fine and coarse moduli spaces are unique up to isomorphism by standard abstract category theory arguments.

2.3.1 The Quot Scheme

One of the most important moduli spaces is the Hilbert scheme or more generally the Quot scheme constructed by Grothendieck in [Gro95]. Many other moduli spaces are constructed as quotient of some Quot scheme with respect to some group action. We refer to [Nit05] as a more modern reference that includes key ideas of Mumford to simplify the proof.

Definition 2.3.2. Let $E \in \text{Coh}(X)$ for a projective scheme $X \in \underline{\text{Sch}}$, $p \in \mathbb{Z}[t]$, $S \in \underline{\text{Sch}}$ and H be a very ample line bundle on X . By E_S we denote the pullback of E to the product $X \times S$ via the first projection. Recall that the *Hilbert polynomial* of coherent sheaf F is defined as the Euler characteristic $p_F(m) = \chi(F \otimes \mathcal{O}(mH))$.

- (i) A *flat family of quotients* of E over S with Hilbert polynomial p is a surjective morphism $E_S \twoheadrightarrow F$ such that F is flat over S , the schematic support of F is proper over S and for any $s \in S$ the restriction of F to the fiber of s has Hilbert polynomial p with respect to H .

(ii) Two families of quotients of E are *equivalent* if their kernels are identical.

For any morphism $T \rightarrow S$ in $\underline{\text{Sch}}$ there is a pullback of families of quotients of E , since all properties are preserved by base change. Therefore, we get the functor $\text{Quot}_{E,H,p} : \underline{\text{Sch}} \rightarrow \underline{\text{Set}}$ that maps $S \in \underline{\text{Sch}}$ to the set of equivalence classes of flat families of quotients of E over S with Hilbert polynomial p with respect to H .

Theorem 2.3.3 ([Gro95]). *Let $E \in \text{Coh}(X)$ for a projective $X \in \underline{\text{Sch}}$, $p \in \mathbb{Z}[t]$ be a polynomial and H be a very ample line bundle on X . Then the functor $\text{Quot}_{E,H,p}$ is represented by a projective scheme $\text{Quot}_{E,H,p} \in \underline{\text{Sch}}$.*

In the case $E = \mathcal{O}_X$ one writes $\mathcal{Hilb}_{H,p} = \text{Quot}_{\mathcal{O}_X,H,p}$, $\text{Hilb}_{H,p} = \text{Quot}_{\mathcal{O}_X,H,p}$ and calls it the *Hilbert scheme*. Quotients of \mathcal{O}_X are in one-to-one correspondence with closed subschemes of X . Therefore, the Hilbert scheme parametrizes closed subschemes of a given scheme X . In Chapter 5 we will study examples of Hilbert schemes of curves in \mathbb{P}^3 .

2.3.2 Moduli Spaces of Semistable Sheaves

Moduli spaces of semistable sheaves were first constructed by Gieseker in [Gie77]. In this section we summarize the treatment in [HL10, Chapter 4]. In order to describe the construction of a moduli space of semistable sheaves, we need to refine the notion of slope semistability for sheaves from the end of subsection 2.2.1. Notice that in all cases where the results of this section are used in this thesis there is no difference.

Let H be a very ample line bundle on a projective scheme $X \in \underline{\text{Sch}}$ and $p \in \mathbb{Z}[t]$ be a polynomial. Let E be a coherent sheaf on X and d be the dimension of the support of E . Due to [HL10, Proof of Lemma 1.2.1] there are rational numbers α_i

for $i = 1, \dots, d$ such that

$$p_E(m) = \sum_{i=0}^d \alpha_i(E) \frac{m^i}{i!}.$$

If X is a variety and E is supported on all of X , then $\alpha_d(E)$ is just the rank of E .

Definition 2.3.4. (i) The *reduced Hilbert polynomial* of a coherent sheaf E is given by

$$\bar{p}_E(m) = \frac{\chi(E \otimes \mathcal{O}(mH))}{\alpha_d(E)}.$$

(ii) A pure coherent sheaf E is called H -Gieseker (semi)stable if for all non trivial proper subsheaves $F \hookrightarrow E$ the inequality

$$\bar{p}_F(m) < (\leq) \bar{p}_E(m)$$

holds for all $m \gg 0$.

(iii) A *flat family of H -Gieseker semistable objects* over a scheme $S \in \underline{\text{Sch}}$ with Hilbert polynomial p is a coherent sheaf $\mathcal{E} \in \text{Coh}(X \times S)$ that is flat over S and for any $s \in S$ the restriction of \mathcal{E} to the fiber of s is an H -Gieseker semistable sheaf with Hilbert polynomial p .

(iv) Two flat families of H -Gieseker semistable objects over a scheme $S \in \underline{\text{Sch}}$ are considered *equivalent* if they differ by tensoring with a line bundle pulled back from S .

If there is a morphism $T \rightarrow S$ in $\underline{\text{Sch}}$, then we can pullback families of H -Gieseker semistable objects from S to T without any issues. Therefore, there is a contravariant functor $\mathcal{M}_{X,p} : \underline{\text{Sch}} \rightarrow \underline{\text{Set}}$ that maps a scheme S to the set of all flat families of H -Gieseker semistable objects over S with Hilbert polynomial p modulo equivalence.

Theorem 2.3.5 ([HL10, Theorem 4.3.4]). *Let $X \in \underline{\text{Sch}}$ be a projective scheme and $p \in \mathbb{Z}[t]$. Then the functor $\mathcal{M}_{X,p}$ is coarsely represented by a projective scheme $M_{X,p}$. If all semistable sheaves are stable, then the moduli space is fine.*

The Hirzebruch-Riemann-Roch Theorem implies that for smooth varieties the following diagram of implications holds.

$$\begin{aligned} \text{Slope Stability} &\Rightarrow \text{Gieseker Stability} \\ &\Rightarrow \text{Gieseker Semistability} \\ &\Rightarrow \text{Slope Semistability} \end{aligned}$$

This means we can apply the above Theorem to construct moduli spaces of slope semistable sheaves in case there are no strictly slope semistable sheaves.

Moreover, the Hirzebruch-Riemann-Roch Theorem also implies that for smooth varieties the Chern character of a coherent sheaf and its Hilbert polynomial determine each other. We will usually fix the Chern character when discussing moduli spaces of semistable sheaves, since it directly connects with various definitions of stability.

2.3.3 Moduli Spaces of Stable Objects in the Derived Category

Much less is known about the properties of moduli spaces for tilt stability or Bridgeland stability on smooth projective threefolds. Beyond special cases, projectivity of moduli spaces of Bridgeland stable objects is a big open question even in the case of surfaces.

Let X be a smooth projective variety over \mathbb{C} , H an ample divisor on X , $v \in K_0(X)$, $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}$ and $s \in \mathbb{R}_{>0}$. The moduli space of tilt semistable objects with respect to α and β with Chern Character $\pm v$ is denoted by $M_{\alpha,\beta}^{\text{tilt}}(v)$. The moduli space of

Bridgeland semistable objects with respect to α , β and s with Chern Character $\pm v$ is denoted by $M_{\alpha,\beta,s}(v)$. We have to define the functors and what a flat family is. Be aware that we generally do not know whether they are represented by a scheme.

Definition 2.3.6. Let $\mathcal{A} \subset D^b(X)$ be the heart of a bounded t-structure.

- (i) A complex $\mathcal{E} \in D^b(X \times S)$ is called S -perfect if it is, locally over S , isomorphic to a bounded complex of flat sheaves over S .
- (ii) A *flat family* of objects in \mathcal{A} over a scheme $S \in \underline{\text{Sch}}$ with Chern character v is an S -perfect complex $\mathcal{E} \in D^b(X \times S)$ such that the derived restriction of \mathcal{E} to any fiber over S is contained in \mathcal{A} and has Chern character v .
- (iii) Two flat families of objects in \mathcal{A} over a scheme $S \in \underline{\text{Sch}}$ are considered *equivalent* if they only differ by tensoring with a line bundle from S .

The condition of being S -perfect is necessary for the derived restriction to be well defined when S is not a smooth variety. The functor $\mathcal{M}_{\alpha,\beta}^{\text{tilt}}(v) : \underline{\text{Sch}} \rightarrow \underline{\text{Set}}$ maps a scheme S to the set of flat families of $\nu_{\alpha,\beta}$ -semistable objects in $\text{Coh}^\beta(X)$ over a scheme $S \in \underline{\text{Sch}}$ with Chern character v . Similarly, the functor $\mathcal{M}_{\alpha,\beta,s}(v) : \underline{\text{Sch}} \rightarrow \underline{\text{Set}}$ maps a scheme S to the set of flat families of $\lambda_{\alpha,\beta,s}$ -semistable objects in $\mathcal{A}^{\alpha,\beta}(X)$ over a scheme $S \in \underline{\text{Sch}}$ with Chern character v .

Assume that $\dim X = 3$. Then the best result towards the properties of these moduli spaces is the following theorem by Piyaratne and Toda.

Theorem 2.3.7 ([PT15]). *Let $v \in K_0(X)$, $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}$ and $s \in \mathbb{R}_{>0}$. Assume that the conjectural BMT-inequality (Conjecture 2.2.12) holds for X . Then the fine moduli space $M_{\alpha,\beta,s}(v)$ is a universally closed algebraic stack of finite type over \mathbb{C} .*

Moreover, if all semistable objects are stable, there is a coarse moduli space that is a proper algebraic space over \mathbb{C} .

The coarse moduli space in the case that all semistable objects are stable becomes fine if the functor is modified to map to families up to equivalence.

No general results about $M_{\alpha,\beta}^{\text{tilt}}(v)$ are known. However, in Section 4.1 we will show that this moduli space is sometimes isomorphic to a moduli space of Bridgeland stable objects and the theorem by Piyaratne and Toda applies. At present it is not known whether these moduli stacks can always be coarsely represented by a proper algebraic space.

We will barely use this general result and only in the case where we have smooth algebraic spaces. The category of smooth proper algebraic spaces over \mathbb{C} is equivalent to the category of compact Moishezon manifolds due to [Knu71]. Therefore, it is possible to escape the technicalities of stacks and algebraic spaces and just deal with complex manifolds in this thesis.

2.3.4 Moduli Spaces of Quiver Representations

A way of showing projectivity of Bridgeland moduli spaces is to show that they are isomorphic to certain moduli spaces of quiver representations. This technique was for example successfully applied in the case of \mathbb{P}^2 in [ABCH13]. We will recall some important facts about moduli spaces of quiver representations. They were introduced in [Kin94].

Definition 2.3.8. (i) A *quiver* Q consists of a finite set Q_0 called the *vertices*, a finite set Q_1 called the *arrows* and two maps $s, t : Q_1 \rightarrow Q_0$ called the *source* and *target* of arrows.

- (ii) A *path* p of length $n \in \mathbb{N}$ in a quiver Q is a sequence of arrows $(a_1, a_2, \dots, a_n) \in Q_1^n$ such that $t(a_i) = s(a_{i+1})$ for all $i = 1, \dots, n - 1$.
- (iii) A *quiver with relations* is a quiver Q plus a finite set of linear combinations of paths in Q .
- (iv) A *complex representation* V of a quiver with relations is a set of complex vector spaces V_x for each $x \in Q_0$ and a set of linear maps $\varphi_{V,a} : V_{s(a)} \rightarrow V_{t(a)}$ for each $a \in Q_1$ such that for these linear maps all the linear combinations given by the relations vanish. The vector $\underline{\dim} V = (\dim V_x)_{x \in Q_0}$ is called the *dimension vector* of V .
- (v) A morphism between representations V and W is a set of linear maps $f_x : V_x \rightarrow W_x$ for all $x \in Q_0$ with commutative diagrams as follows for all $a \in Q_1$

$$\begin{array}{ccc}
 V_{s(a)} & \xrightarrow{\varphi_{V,a}} & V_{t(a)} \\
 \downarrow f_{s(a)} & & \downarrow f_{t(a)} \\
 W_{s(a)} & \xrightarrow{\varphi_{W,a}} & W_{t(a)}.
 \end{array}$$

It is well known that the category of quiver representations is equivalent to the category of representations over a finite dimensional algebra, making it an abelian category. In order to study moduli spaces of quiver representations it is necessary to define a notion of stability for them. We will use the slightly modified definition found in [Rei08].

Let Q be a quiver with relations and $\theta : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ a linear map. We define a central charge by

$$\mu_\theta(V) = \frac{\theta(\underline{\dim} V)}{\dim V},$$

where V is a representation of Q and $\dim V = \sum_{x \in V} \dim V_x$. As in the geometric context a representation V is called θ -(semi)stable if for all proper non trivial subrepresentations $W \hookrightarrow V$ the inequality $\mu_\theta(W) < (\leq) \mu_\theta(V)$ holds.

Definition 2.3.9. (i) A *family of representations* of a quiver with relations Q over a scheme $S \in \underline{\text{Sch}}$ is a set of locally free sheaves $V_x \in \text{Coh}(S)$ for each $x \in Q_0$ and a set of morphisms of coherent sheaves $\varphi_{V,a} : V_{s(a)} \rightarrow V_{t(a)}$ satisfying the relations of Q .

(ii) Two families of representations of a quiver with relations Q over a scheme $S \in \underline{\text{Sch}}$ are considered *equivalent* if they only differ by tensoring all locally free sheaves V_x with the same line bundle coming from S .

Fix a dimension vector $v \in \mathbb{N}^{Q_0}$ and a linear map $\theta : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$. Similarly to the previous cases, there is a functor $\mathcal{R}_{Q,v,\theta} : \underline{\text{Sch}} \rightarrow \underline{\text{Set}}$ that maps a scheme $S \in \underline{\text{Sch}}$ to the set of all families of representations of Q over S with dimension vector v such that the restriction to any $s \in S$ is a θ -semistable representation up to equivalence.

Theorem 2.3.10 ([Kin94]). *Let Q be a quiver with relations, $v \in \mathbb{N}^{Q_0}$ and $\theta : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ a linear map. Then the functor $\mathcal{R}_{Q,v,\theta}$ is coarsely represented by a projective scheme $R_{Q,v,\theta}$ over \mathbb{C} . Moreover, if all θ -semistable representations with dimension vector v are θ -stable the moduli space is fine.*

2.3.5 Deformation Theory

Throughout this thesis it will be necessary to compute the tangent spaces of certain moduli spaces. This is usually done through means of deformation theory. In this subsection we will recall the important facts needed. As always let X be a smooth projective variety over the complex numbers.

Theorem 2.3.11 ([HL10, Theorem 4.5.2]). *Fix a Hilbert polynomial $p \in \mathbb{Z}[t]$ and let $M_{X,p}$ be the moduli spaces of Gieseker semistable sheaves with Hilbert polynomial p . If E is a stable coherent sheaf, then its Zariski tangent space in $M_{X,p}$ is given by $\text{Ext}^1(E, E)$.*

The deformation theory for objects in the derived category has been worked out in [Ina02] and [Lie06]. In particular, the above result will also hold for stable objects in Bridgeland stability. More precisely, let σ be a Bridgeland stability condition on X . If $E \in D^b(X)$ is a σ -stable object, then the Zariski tangent space of the corresponding point in the moduli space of Bridgeland stable objects is given by $\text{Ext}^1(E, E)$.

Chapter 3: The Bayer-Macri-Toda Inequality

The main result of this chapter is the following theorem. It was published in [Sch14].

Theorem 3.0.1. *The BMT-inequality holds for the smooth quadric threefold Q , i.e. any $\nu_{\alpha,\beta}$ -semistable object $E \in \text{Coh}^\beta(Q)$ satisfies*

$$\alpha^2 Q^{\text{tilt}}(E) + 4(H \cdot \text{ch}_2^\beta(E))^2 - 6(H^2 \cdot \text{ch}_1^\beta(E)) \text{ch}_3^\beta(E) \geq 0,$$

where H is the ample generator of $\text{Pic}(Q)$.

We will improve upon the proof in a couple of places over what was done in [Sch14], but the general idea remains unchanged.

3.1 The Quadric Threefold

In order to prove the BMT-inequality (Conjecture 2.2.12) for the smooth quadric threefold Q , we need to recall some facts about its bounded derived category of coherent sheaves $D^b(Q)$. In the following we view Q as being cut out by the equation $x_0^2 + x_1x_2 + x_3x_4 = 0$ in \mathbb{P}^4 .

Since the open subvariety of Q defined by $x_1 \neq 0$ is isomorphic to \mathbb{A}^3 , the Picard group of Q is isomorphic to \mathbb{Z} and is generated by a very ample line bundle $\mathcal{O}(H)$.

Moreover, the equality $H^3 = 2$ holds because a general line in \mathbb{P}^4 intersects Q in two points.

Definition 3.1.1. A *strong exceptional collection* is a sequence E_1, \dots, E_r of objects in $D^b(X)$ such that $\text{Ext}^i(E_l, E_j) = 0$ for all l, j and $i \neq 0$, $\text{Hom}(E_j, E_j) = \mathbb{C}$ and $\text{Hom}(E_l, E_j) = 0$ for all $l > j$. Moreover, it is called *full* if E_1, \dots, E_r generate $D^b(X)$ via shifts and extensions.

On Q line bundles are not enough to obtain a full strong exceptional collection. Therefore, we need to introduce the spinor bundle S . We refer to [Ott88] for a more detailed treatment. The spinor bundle is defined via an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \rightarrow \iota_* S \rightarrow 0$$

where $\iota : Q \hookrightarrow \mathbb{P}^4$ is the inclusion and the first map is given by a matrix M such that $M^2 = (x_0^2 + x_1x_2 + x_3x_4)I_4$ for the identity 4×4 matrix I_4 . Restricting the second morphism to Q leads to

$$0 \rightarrow S(-1) \rightarrow \mathcal{O}_Q^{\oplus 4} \rightarrow S \rightarrow 0. \quad (3.1)$$

Due to Kapranov (see [Kap88])

$$\mathcal{O}(-1), S(-1), \mathcal{O}, \mathcal{O}(1)$$

is a strong full exceptional collection on $D^b(Q)$.

Explicit computations lead to a resolution of the skyscraper sheaf \mathcal{O}_x given by

$$0 \rightarrow \mathcal{O}(-1) \rightarrow S(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus 4} \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_x \rightarrow 0 \quad (3.2)$$

for any $x \in Q$.

3.2 Limit Stability

In order to connect the strong exceptional collection on Q with Bridgeland stability, it is necessary to reduce the values of α and β for which we have to prove the BMT-conjecture. It is not necessary to restrict to Q for this, so let X be a smooth projective threefold with an ample line bundle H throughout this section.

Lemma 3.2.1 ([BMS14, Theorem 4.2]). *The BMT-inequality holds for all $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ if and only if for any $\nu_{\alpha,\beta}$ -stable object $E \in \text{Coh}^\beta(X)$ with $\nu_{\alpha,\beta}(E) = 0$ the inequality*

$$\text{ch}_3^\beta(E) \leq \frac{\alpha^2}{6} H^2 \cdot \text{ch}_1^\beta(E)$$

holds.

Proof. Assume that $E \in \text{Coh}^\beta(Q)$ is $\nu_{\alpha,\beta}$ -stable with $\nu_{\alpha,\beta}(E) = 0$. Then

$$H \cdot \text{ch}_2^\beta(E) = \frac{\alpha^2}{2} H^3 \cdot \text{ch}_0^\beta$$

holds. Moreover, we must have $H \cdot \text{ch}_1^\beta(E) > 0$. From this it is immediate that the BMT-inequality and the inequality $\text{ch}_3^\beta(E) \leq \frac{\alpha^2}{6} H^2 \cdot \text{ch}_1^\beta(E)$ are equivalent to each other on a completely numerical basis. The crucial part of the proof is to show that one can reduce the general conjecture to this special case.

The set of (α, β) satisfying $Q_{\alpha,\beta}(E) = 0$ is equal to the set of (α, β) such that $\nu_{\alpha,\beta}(E) = \nu_{\alpha,\beta}(\text{ch}_1(E), 2 \text{ch}_2(E), 3 \text{ch}_3(E), 0)$. Therefore, it is a numerical wall in tilt stability if it is non empty. By the structure theorem for walls in tilt stability, numerical walls do not intersect. This implies both that stable objects stay stable along numerical walls and also that the BMT-inequality is fulfilled at all points along a numerical wall or at none. Any point (α, β) lies on a unique numerical wall. All

numerical walls intersect the hyperbola $\nu_{\alpha,\beta}(E) = 0$ except the unique vertical wall that only exists if $\text{ch}_0(E) \neq 0$.

Assume (α, β) is a point on the unique numerical vertical for a stable $E \in \text{Coh}^\beta(X)$. By the structure theorem for walls in tilt stability this is given by the equation

$$\beta = \frac{\text{ch}_1(E)}{\text{ch}_0(E)}.$$

Moreover, any tilt semistable object satisfies $Q^{\text{tilt}}(E) \geq 0$ by the Bogomolov inequality. Under these circumstances the BMT-inequality is true. \square

Next we want to move down the hyperbola and show that it is enough to show the conjecture for small values of α .

Lemma 3.2.2 ([Mac14b, Proposition 2.7]). *The BMT-conjecture holds for all $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ if and only if there is $\varepsilon > 0$ such that it holds for all objects $E \in \text{Coh}^\beta(X)$ with $\nu_{\alpha,\beta}(E) = 0$ and $\alpha < \varepsilon$.*

Proof. Let $E \in \text{Coh}^{\beta_0}(X)$ be a ν_{α_0,β_0} semistable object. By Lemma 3.2.1 we can assume that $\nu_{\alpha_0,\beta_0}(E) = 0$. The proof will proceed by induction on the integer $Q^{\text{tilt}}(E)$.

Assume $Q^{\text{tilt}}(E) = 0$ holds. By Lemma 2.2.5, E cannot be destabilized along the hyperbola $\nu_{\alpha,\beta}(E) = 0$. By assumption we know that the conjecture holds for $\alpha < \varepsilon$, i.e.

$$\text{ch}_3^\beta(E) \leq \frac{a^2}{6} H^2 \text{ch}_1^\beta(E).$$

Since $Q_{\alpha,\beta}(E) \geq 0$ describes the complement of a semidisc with center on the β -axis, this inequality will become stronger when decreasing along the hyperbola. That finishes this case.

Assume next that $Q^{\text{tilt}}(E) > 0$. If E remains semistable all the way down the hyperbola the same argument as for $Q^{\text{tilt}}(E) = 0$ finishes the proof. If not, there is a wall at which E destabilizes. Let (α, β) be the point of intersection of the wall and the hyperbola and let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be a sequence of semistable objects defining the wall. By induction both F and G satisfy the BMT inequality and additivity of the Chern character finishes the proof. \square

The reader might have noticed that the argument in Lemma 3.2.1 would allow to reduce to small values of α by moving along the semi circular walls. However, this is deceiving as the equivalence of the BMT conjecture and the construction of Bridgeland stability in Theorem 2.2.20 uses the version on the hyperbola. We will make use of this equivalence in the proof.

Lastly, we will reduce the values of β for which the conjecture has to be proven.

Lemma 3.2.3 ([Mac14b, Lemma 3.2]). *The BMT-conjecture holds for all $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ if and only if there is $\varepsilon > 0$ such that it holds for all objects $E \in \text{Coh}^\beta(X)$ with $\nu_{\alpha,\beta}(E) = 0$, $\alpha < \varepsilon$ and $\beta \in [-\frac{1}{2}, 0]$.*

Proof. The reduction to $\alpha < \varepsilon$ was done in the previous Lemma 3.2.2. Since $\text{ch}^\beta(E) = \text{ch}^{\beta+n}(E \otimes \mathcal{O}(nH))$ for any $E \in D^b(X)$ and $n \in \mathbb{Z}$, we can use tensoring by line bundles to reduce to the case $\beta \in [-1/2, 1/2)$.

Assume $E \in \text{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$ -semistable for $\beta \in (0, \frac{1}{2})$ with $\nu_{\alpha,\beta}(E) = 0$. By Proposition 2.2.15 there is a $\nu_{\alpha,-\beta}$ -semistable object $\tilde{E} \in \text{Coh}^{-\beta}(X)$ and a sheaf T supported in dimension 0 together with a triangle

$$\tilde{E} \rightarrow \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)[1] \rightarrow T[-1] \rightarrow \tilde{E}[1].$$

We have

$$\mathrm{ch}^{-\beta}(\tilde{E}) = (-\mathrm{ch}_0^\beta(E), \mathrm{ch}_1^\beta(E), -\mathrm{ch}_2^\beta(E), \mathrm{ch}_3^\beta(E) + \mathrm{ch}_3(T)).$$

In particular, $\nu_{\alpha,-\beta}(\tilde{E}) = 0$ and

$$Q_{\alpha,\beta}(E) = Q_{\alpha,-\beta}(\tilde{E}) + 6 \mathrm{ch}_1^\beta(E) \mathrm{ch}_3(T)$$

hold. Since $\mathrm{ch}_3(T) \geq 0$, we have reduced to the case $\beta \in [-\frac{1}{2}, 0]$. \square

3.3 Inequality on the Quadric Threefold

In this section the BMT-inequality will be proven for objects $\nu_{\alpha,\beta}$ -semistable objects $E \in \mathrm{Coh}^\beta(X)$ where $\alpha < \frac{1}{3}$ and $\beta \in [-\frac{1}{2}, 0]$. Due to Lemma 3.2.3 this is enough to prove Theorem 3.0.1. The following technical proposition provides the basis of the proof.

Proposition 3.3.1 ([BMT14, Lemma 8.1.1]). *Let $\mathcal{C} \subset D^b(X)$ be the heart of a bounded t -structure with the following properties.*

(i) *There exists $\phi_0 \in (0, 1)$ and $s_0 \in \mathbb{Q}$ such that*

$$Z_{\alpha,\beta,s_0}(\mathbb{C}) \subset \{re^{\pi\phi i} : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1\}.$$

(ii) *The inclusion $\mathcal{C} \subset \langle \mathcal{A}^{\alpha,\beta}(X), \mathcal{A}^{\alpha,\beta}(X)[1] \rangle$ holds.*

(iii) *For all points $x \in X$ we have $\mathcal{O}_x \in \mathcal{C}$ and for all proper subobjects $C \hookrightarrow \mathcal{O}_x$ in \mathcal{C} the inequality $\Im Z_{\alpha,\beta,s_0}(C) > 0$ holds.*

Then the pair $(Z_{\alpha,\beta,s}, \mathcal{A}^{\alpha,\beta}(X))$ is a stability condition on $D^b(X)$ for all $s > s_0$.

In the next chapter we will need a slightly stronger connection between \mathcal{C} and $\mathcal{A}^{\alpha,\beta}(X)$ in the case $X = \mathbb{P}^3$.

Corollary 3.3.2. *Assume that notation and properties of Proposition 3.3.1 hold. Let S be the set of $s > s_0$ such that there is $\phi_s \in (0, 1]$ satisfying*

$$Z_{\alpha,\beta,s}(\mathbb{C}) \subset \{re^{\pi\phi i} : r > 0, \phi_s < \phi \leq \phi_s + 1\}. \quad (3.3)$$

For any $s \in S$ we denote the slicing of the stability condition $(Z_{\alpha,\beta,s}, \mathcal{A}^{\alpha,\beta}(X))$ by $P_{\alpha,\beta,s}$. Then $\mathcal{C} = P_{\alpha,\beta,s}((\phi_s, \phi_s + 1])$ holds.

Proof. Let $\mathcal{T} = \mathcal{A}^{\alpha,\beta}(X) \cap \mathcal{C}$ and $\mathcal{F} = \mathcal{A}^{\alpha,\beta}(X) \cap \mathcal{C}[-1]$. Then the equality $\mathcal{C} = \langle \mathcal{T}, \mathcal{F}[1] \rangle$ holds.

Let $E \in \mathcal{A}^{\alpha,\beta}(X)$ be $Z_{\alpha,\beta,s}$ -stable. We will prove that E is in \mathcal{T} or \mathcal{F} depending on its phase. There are unique $T \in \mathcal{T}$ and $F \in \mathcal{F}$ together with an exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0.$$

Assume $\phi_{\alpha,\beta,s}(E) > \phi_s$. If $F \neq 0$ holds, then by stability of E we get $\phi_{\alpha,\beta,s}(E) < \phi_{\alpha,\beta,s}(F)$. Due to (3.3) this contradicts $F \in \mathcal{C}[-1]$. Thus, $E = T \in \mathcal{T}$ holds.

Assume $\phi_{\alpha,\beta,s}(E) \leq \phi_s$. If $T \neq 0$ holds, then stability of E implies $\phi_{\alpha,\beta,s}(E) > \phi_{\alpha,\beta,s}(T)$. Because of (3.3) this contradicts $T \in \mathcal{C}$. Therefore, $E = F \in \mathcal{F}$ holds. By the existence and uniqueness of Harder-Narasimhan filtrations both \mathcal{F} and \mathcal{T} are generated by stable objects. This finishes the proof. \square

Due to [Bon89] a full strong exceptional collection induces an equivalence between $D^b(X)$ and the bounded derived category of finitely generated modules over some finite dimensional algebra A .

Theorem 3.3.3. *Let E_0, \dots, E_n be a strong full exceptional collection on $D^b(X)$, $A := \text{End}(\bigoplus E_i)$ and $\text{mod } -A$ be the category of right A -modules of finite rank. Then*

the functor

$$R\mathrm{Hom}(A, \cdot) : D^b(X) \rightarrow D^b(\mathrm{mod} - A)$$

is an exact equivalence. Under this identification the E_i correspond to the indecomposable projective A -modules.

In the special case of the smooth quadric threefold Q , we get the heart of a bounded t-structure by setting

$$\mathcal{C} := \langle \mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1) \rangle.$$

The category \mathcal{C} is isomorphic to the category of finitely generated modules over some finite dimensional algebra A and $\mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1)$ are the simple objects. There is a way to determine the projective objects out of these, but we will not need this.

We will show that the conditions of Proposition 3.3.1 are fulfilled for this \mathcal{C} and $s_0 = \frac{1}{6}$. By using (3.1) we can obtain the following lemma.

Lemma 3.3.4. *For all $n \in \mathbb{N}$ we have*

$$\mathrm{ch}^\beta(\mathcal{O}(n)) = (1, (n - \beta)H, (n - \beta)^2 \frac{H^2}{2}, \frac{1}{3}(n - \beta)^3).$$

The Chern character of $S(-1)$ is given by

$$\mathrm{ch}^\beta(S(-1)) = (2, -(2\beta + 1)H, \beta(\beta + 1)H^2, \frac{1}{6} - \beta^2 - \frac{2}{3}\beta^3).$$

We have the following μ -slopes

$$\begin{aligned} \mu_\beta(\mathcal{O}(1)) &= 1 - \beta, & \mu_\beta(\mathcal{O}) &= -\beta, \\ \mu_\beta(\mathcal{O}(-1)) &= -\beta - 1, & \mu_\beta(S(-1)) &= -\beta - \frac{1}{2}. \end{aligned}$$

The ν -slopes for the same sheaves are given by

$$\begin{aligned}\nu_{\alpha,\beta}(\mathcal{O}(1)) &= \frac{(1-\beta)^2 - \alpha^2}{2(1-\beta)}, & \nu_{\alpha,\beta}(\mathcal{O}) &= \frac{\alpha^2 - \beta^2}{2\beta}, \\ \nu_{\alpha,\beta}(\mathcal{O}(-1)) &= \frac{\alpha^2 - (1+\beta)^2}{2(1+\beta)}, & \nu_{\alpha,\beta}(S(-1)) &= \frac{\alpha^2 - \beta(\beta+1)}{2\beta+1}.\end{aligned}$$

Finally, the Z values can be computed as

$$\begin{aligned}Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}(1)) &= \frac{1}{3}((1-\beta)^2 - \alpha^2)(\beta - 1 + 3i), \\ Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}) &= \frac{1}{3}(\beta^2 - \alpha^2)(\beta + 3i), \\ Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}(-1)) &= \frac{1}{3}((1+\beta)^2 - \alpha^2)(\beta + 1 + 3i), \\ Z_{\alpha,\beta,\frac{1}{6}}(S(-1)) &= \frac{1}{6}(2\beta+1)(2\beta^2 + 2\beta - 1 - 2\alpha^2) + 2i(\beta^2 + \beta - \alpha^2).\end{aligned}$$

At this point we can prove the first assumption in Proposition 3.3.1.

Lemma 3.3.5. *There exists $\phi_0 \in (0, 1)$ such that*

$$Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{C}) \subset \{re^{\pi\phi i} : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1\}.$$

Proof. It suffices to show that the four generators of \mathcal{C} are contained in some half plane of \mathbb{C} . There are two different cases to deal with.

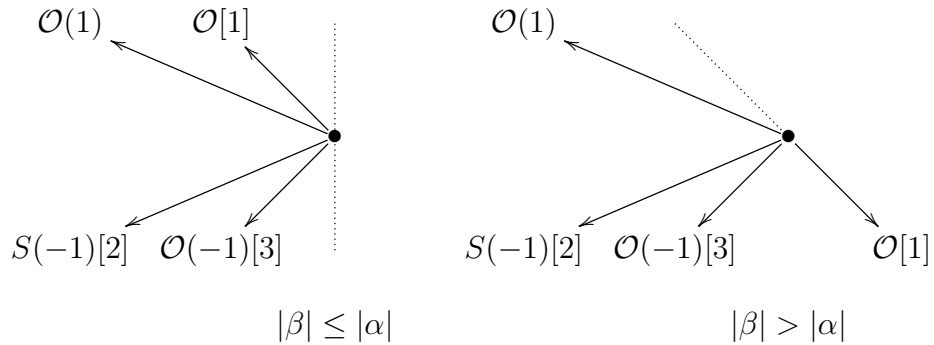


Figure 3.1: Central charge for exceptional objects

Lemma 3.3.4 shows that the half plane of points with negative real part works if $|\beta| \leq |\alpha|$. The half plane left of the line through 0 and $Z_{\alpha,\beta,\frac{1}{6}}(\mathcal{O}[1])$ works in the case $|\beta| > |\alpha|$. Figure 3.1 shows the $Z_{\alpha,\beta,\frac{1}{6}}$ values. \square

The next lemma shows that condition (ii) in Proposition 3.3.1 holds.

Lemma 3.3.6. *The inclusion $\mathcal{C} \subset \langle \mathcal{A}^{\alpha,\beta}(Q), \mathcal{A}^{\alpha,\beta}(Q)[1] \rangle$ hold for all $\alpha \in (0, \frac{1}{3})$ and $\beta \in [-1/2, 0]$.*

Proof. If $L[i] \in \text{Coh}^\beta(Q)$ holds for a line bundle L and $i \in \{0, 1\}$, then $L[i]$ is tilt-stable. This was originally proven in [BMT14, Proposition 7.4.1], but we will proof a slightly more general statement in Proposition 4.2.1. By Lemma 3.3.4 we get immediately $\mathcal{O}(-1)[3], \mathcal{O}[1], \mathcal{O}(1) \in \langle \mathcal{A}^{\alpha,\beta}(Q), \mathcal{A}^{\alpha,\beta}(Q)[1] \rangle$.

By [Ott88] the spinor bundle $S(-1)$ is μ -stable. The inequality $\mu_{\alpha,\beta}(S(-1)) \leq 0$ leads to $S(-1)[1] \in \text{Coh}^\beta(Q)$. We have to show that $S(-1)[1]$ is $\nu_{\alpha,\beta}$ -semistable in our range.

The unique vertical numerical wall for $S(-1)$ in tilt stability occurs at $\beta = -1/2$. The hyperbola $\nu_{\alpha,\beta}(S(-1)) = 0$ intersects the β -axis at $\beta = -1$ and $\beta = 0$. Moreover, we have $\text{ch}_1(S(-1)[1]) = H$. Therefore, $S(-1)[1]$ is semistable along $\beta = 0$ due to Lemma 2.2.13. However, by the structure of walls in tilt stability $S(-1)[1]$ can only be unstable in our range for α and β if it destabilizes along $\beta = 0$. \square

The proof of Theorem 3.0.1 can be concluded by the next lemma.

Lemma 3.3.7. *For all $x \in X$, we have $\mathcal{O}_x \in \mathcal{C}$ and for all proper subobjects $C \hookrightarrow \mathcal{O}_x$ in \mathcal{C} the inequality $\Im Z_{\alpha,\beta,\frac{1}{6}}(C) > 0$ holds.*

Proof. We have $\mathcal{O}_x \in \mathcal{C}$ because of the resolution in (3.2)

$$0 \rightarrow \mathcal{O}(-1) \rightarrow S(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus 4} \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_x \rightarrow 0.$$

For the second assertion we need to figure out which are the subobjects of $\mathcal{O}_x \in \mathcal{C}$.

Any object in \mathcal{C} is given by a complex F of the form

$$0 \rightarrow \mathcal{O}(-1)^{\oplus a} \rightarrow S(-1)^{\oplus b} \rightarrow \mathcal{O}^{\oplus c} \rightarrow \mathcal{O}(1)^{\oplus d} \rightarrow 0.$$

for $a, b, c, d \in \mathbb{Z}_{\geq 0}$. Since \mathcal{C} is the category of representations of a quiver with relations with simple objects $\mathcal{O}(-1)[3], S(-1)[2], \mathcal{O}[1], \mathcal{O}(1)$, we can interpret $v(F) = (a, b, c, d)$ as the dimension vector of that representation. Therefore, $F \hookrightarrow \mathcal{O}_x \twoheadrightarrow G$ implies $a \leq 1, b \leq 2, c \leq 4$ and $d \leq 1$. If F is non trivial, then there is a simple object $T_1 \hookrightarrow F$. But the only simple object with non trivial morphism into \mathcal{O}_x is $\mathcal{O}(1)$. Therefore, the equality $d = 1$ holds. If G is non trivial, then there exists a simple quotient $\mathcal{O}_x \twoheadrightarrow T_2$. By Serre duality, the only simple quotient is $T_2 = \mathcal{O}(-1)[3]$. That implies $a = 0$. Assume $b = 2$, but $c < 4$. Then we obtain $G = \mathcal{O}(-1)[3] \oplus \mathcal{O}[1]^{\oplus 4-c} \twoheadrightarrow \mathcal{O}[1]$. A contradiction comes from $\text{Hom}(\mathcal{O}_x, \mathcal{O}[1]) = 0$. Therefore, $b = 2$ implies $c = 4$. The remaining cases are

$$v(F) \in \{(0, 2, 4, 1)\} \cup \{(0, b, c, 1) : b \in \{0, 1\}, c \in \{0, 1, 2, 3, 4\}\}.$$

Since $\Im Z_{\alpha, \beta, \frac{1}{6}}(S(-1)) < 0$ holds, the case $b = 0$ follows from $b = 1$. With the same argument $v(F) = (0, 1, 4, 1)$ follows from $v(F) = (0, 2, 4, 1)$. Depending on the sign of $\Im Z_{\alpha, \beta, \frac{1}{6}}(\mathcal{O})$, we can reduce $b = 1$ to either $c = 0$ or $c = 4$. Hence, two cases are left.

$v(F)$	$\Im Z_{\alpha, \beta, \frac{1}{6}}$
$(0, 2, 4, 1)$	$\alpha((1 + \beta)^2 - \alpha^2)$
$(0, 1, 0, 1)$	$\alpha(1 - 3(\beta^2 - \alpha^2))$

Table 3.1: Imaginary part of central charge

For all of them $\mathfrak{S}Z_{\alpha,\beta,\frac{1}{6}}$ is positive (see Table 3.1). \square

The arguments in this chapter can also deal with the case $X = \mathbb{P}^3$ as was done in [Mac14b] and our proof is based on it. The general challenge lies in choosing the correct exceptional collection. For \mathbb{P}^3 one chooses $\mathcal{O}(-1), T(-2), \mathcal{O}, \mathcal{O}(1)$ where T is the tangent bundle.

3.4 Counterexample

In this section, we will present a counterexample to the BMT inequality due to [Sch16]. Let $f : X \rightarrow \mathbb{P}^3$ be the blow up of \mathbb{P}^3 in a point P . The Picard group of X is well known to be a free abelian group with two generators $\mathcal{O}(L) = f^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}(E)$, where $E = f^{-1}(P)$. The variety X is Fano with canonical divisor given by $-4L + 2E$. In particular, $H = 2L - E$ is an ample divisor. We also have the intersection products $L^3 = E^3 = 1$, $L \cdot E = 0$ and $H^3 = 7$. The goal of this section is to prove the following counterexample to the conjectural inequality.

Theorem 3.4.1. *There exists $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that the line bundle $\mathcal{O}_X(L)$ is $\nu_{\alpha,\beta}$ -stable, but $Q_{\alpha,\beta}(\mathcal{O}_X(L)) < 0$.*

Proof. Since $\mathcal{O}(L)$ is a line bundle it is a slope semistable sheaf. In particular, either $\mathcal{O}(L)$ or $\mathcal{O}(L)[1]$ is $\nu_{\alpha,\beta}$ -stable for all $\alpha \gg 0$.

A straightforward computation shows $H^3 \cdot \text{ch}_0(\mathcal{O}(L)) = 7$, $H^2 \cdot \text{ch}_1(\mathcal{O}(L)) = 4$, $H \cdot \text{ch}_2(\mathcal{O}(L)) = 1$ and $\text{ch}_3(\mathcal{O}(L)) = 1/6$. In particular, this means $H^2 \cdot \text{ch}_1^{1/2}(\mathcal{O}(L)) = 1/2$. If $F \hookrightarrow \mathcal{O}(L)$ destabilizes along the line $\beta = 1/2$, we must have $\text{ch}_1^{1/2}(F) \in \{0, 1/2\}$. That means either F or the quotient has slope infinity independently of α , a contradiction.

A completely numerical computation shows that $Q_{\alpha,\beta}(\mathcal{O}(L)) \geq 0$ is equivalent to the inequality

$$\alpha^2 + \left(\beta - \frac{1}{4}\right)^2 \geq \frac{1}{16}.$$

We are done if we can prove that there is no wall with equality in this inequality for $\mathcal{O}(L)$. Assume there is a destabilizing sequence $0 \rightarrow F \rightarrow \mathcal{O}(L) \rightarrow G \rightarrow 0$ giving exactly this wall. Taking the long exact sequence in cohomology, we get $H^3 \cdot \text{ch}_0(F) \geq 7$. By definition of $\text{Coh}^\beta(X)$ we have the inequalities $H^2 \cdot \text{ch}_1^\beta(\mathcal{O}(L)) \geq H^2 \cdot \text{ch}_1^\beta(F) \geq 0$ for all $\beta \in [0, 1/2]$. This can be rewritten as

$$4 + \beta(H^3 \cdot \text{ch}_0(F) - 7) \geq H^2 \cdot \text{ch}_1(F) \geq \beta H^3 \cdot \text{ch}_0(F).$$

Notice that the middle term is independent of β and we can vary β independently on the left and right. Therefore, we get $4 \geq H^2 \cdot \text{ch}_1(F) \geq H^3 \cdot \text{ch}_0(F)/2$. This means $H^2 \cdot \text{ch}_1(F) = 4$ and $H^3 \cdot \text{ch}_0(F) = 7$. This does not give the correct wall. \square

Chapter 4: From Tilt Stability to Bridgeland Stability

In this chapter we prove a theorem that allows to translate computations in tilt stability to wall crossings in Bridgeland stability. Moreover, a theorem about the stability of power of line bundles is proven for \mathbb{P}^3 . As a consequence of both these results we show that, for certain types of sheaves, several wall crossings in Bridgeland stability connect the moduli space of semistable sheaves to a smooth projective variety. The content of this chapter is essentially due to [Sch15].

4.1 The Connection Theorem

We will see in concrete examples in the next chapters that computation of walls in tilt stability are very comparable to those on surfaces and techniques usually transfer. The fundamental issue is the fact that Lemma 2.2.19 is in general incorrect in tilt stability. That makes it difficult to precisely determine how the corresponding sets of stable objects change at a wall. Often times fundamentally different types of exact sequences lead to identical walls due to the fact that the definition of tilt stability disregards the third Chern character.

In Bridgeland stability the situation is reversed. While it is generally difficult to numerically determine walls, it is usually easier to compute the actual effects of wall crossings on the moduli space.

Let $v = (v_0, v_1, v_2, v_3)$ be the Chern character of an object in $D^b(X)$. Recall that for any $\alpha > 0$, $\beta \in \mathbb{R}$ and $s > 0$ the set of $\lambda_{\alpha, \beta, s}$ -semistable objects with Chern character $\pm v$ is denoted by $M_{\alpha, \beta, s}(v)$. Analogous to our notation for twisted Chern characters we write $v^\beta = (v_0^\beta, v_1^\beta, v_2^\beta, v_3^\beta) := v \cdot e^{-\beta H}$. We also write

$$P_v := \{(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R} : \nu_{\alpha, \beta}(v) > 0\}.$$

The goal of this section is to prove the following theorem. Under some hypotheses, it roughly says that on one side of the hyperbola $\{\nu_{\alpha, \beta}(v) = 0\}$ all the chambers and wall crossings of tilt stability occur in a potentially refined way in Bridgeland stability. In general, the difference between these wall crossings and the corresponding situation in tilt stability is comparable to the difference between slope stability and Gieseker stability. Using the theory of polynomial stability conditions from [Bay09] one can define an analogue of that situation to make this precise. We will not do this as we are not aware of any interesting examples in which the difference matters.

Theorem 4.1.1. *Let v be the Chern character of an object in $D^b(X)$, $\alpha_0 > 0$, $\beta_0 \in \mathbb{R}$ and $s > 0$ such that $\nu_{\alpha_0, \beta_0}(v) = 0$ and $H^2 v_1^{\beta_0} > 0$.*

(i) *Assume there is an actual wall in Bridgeland stability for v at (α_0, β_0) given by*

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0.$$

That means $\lambda_{\alpha_0, \beta_0, s}(F) = \lambda_{\alpha_0, \beta_0, s}(G)$ and $\text{ch}(E) = \pm v$ for semistable $E, F, G \in \mathcal{A}^{\alpha_0, \beta_0}(X)$. Further assume there is a neighborhood U of (α_0, β_0) such that the same sequence also defines an actual wall in $U \cap P_v$, i.e. E, F, G remain semistable in $U \cap P_v \cap \{\lambda_{\alpha, \beta, s}(F) = \lambda_{\alpha, \beta, s}(G)\}$. Then $E[-1], F[-1], G[-1] \in \text{Coh}^{\beta_0}(X)$ are ν_{α_0, β_0} -semistable. In particular, there is an actual wall in tilt stability at (α_0, β_0) .

(ii) Assume that all ν_{α_0, β_0} -semistable objects are stable. Then there is a neighborhood U of (α_0, β_0) such that

$$M_{\alpha, \beta, s}(v) = M_{\alpha, \beta}^{\text{tilt}}(v)$$

for all $(\alpha, \beta) \in U \cap P_v$. Moreover, in this case all objects in $M_{\alpha, \beta, s}(v)$ are $\lambda_{\alpha, \beta, s}$ -stable.

(iii) Assume there is a wall in tilt stability intersecting (α_0, β_0) . If the set of tilt stable objects is different on the two sides of the wall, then there is at least one actual wall in Bridgeland stability in P_v that has (α_0, β_0) as a limiting point.

(iv) Assume there is an actual wall in tilt stability for v at (α_0, β_0) given by

$$0 \rightarrow F^n \rightarrow E \rightarrow G^m \rightarrow 0$$

such that $F, G \in \text{Coh}^{\beta_0}(X)$ are ν_{α_0, β_0} -stable objects, $\text{ch}(E) = \pm v$ and there is an equality $\nu_{\alpha_0, \beta_0}(F) = \nu_{\alpha_0, \beta_0}(G)$. Assume further that the set

$$P_v \cap P_{\text{ch}(F)} \cap P_{\text{ch}(G)} \cap \{\lambda_{\alpha, \beta, s}(F) = \lambda_{\alpha, \beta, s}(G)\}$$

is non empty. Then there is a neighborhood U of (α_0, β_0) such that F, G are $\lambda_{\alpha, \beta, s}$ -stable for all $(\alpha, \beta) \in U \cap P_v \cap \{\lambda_{\alpha, \beta, s}(F) = \lambda_{\alpha, \beta, s}(G)\}$. In particular, there is an actual wall in Bridgeland stability restricted to $U \cap P_v$ defined by the same sequence.

Before we can prove this theorem, we need three preparatory lemmas. The following lemma shows how to descend tilt stability on the hyperbola $\{\nu_{\alpha, \beta}(v) = 0\}$ to Bridgeland stability on one side of the hyperbola. The main issue is that the hyperbola can potentially be a wall itself.

Lemma 4.1.2. *Assume $E \in \text{Coh}^{\beta_0}(X)$ is a ν_{α_0, β_0} -stable object such that $\nu_{\alpha_0, \beta_0}(E) = 0$ and fix some $s > 0$. Then $E[1]$ is $\lambda_{\alpha_0, \beta_0, s}$ -semistable. Moreover, there is a neighborhood U of (α_0, β_0) such that E is $\lambda_{\alpha, \beta, s}$ -stable for all $(\alpha, \beta) \in U \cap P_{\text{ch}(E)}$.*

Proof. By definition $E[1] \in \mathcal{A}^{\alpha_0, \beta_0}(X)$. Since $\lambda_{\alpha_0, \beta_0, s}(E[1]) = \infty$, the object $E[1]$ is semistable at this point. Let $E[1] \rightarrow G$ be a stable factor in a Jordan-Hölder filtration. There is a neighborhood U of (α_0, β_0) such that any destabilizing stable quotient of E in $U \cap P_{\text{ch}(E)}$ is of this form. This can be done since there is a locally finite wall and chamber structure such that the Harder-Narasimhan filtration of E is constant in each chamber. Let F be the kernel of this quotient, i.e. there is an exact sequence $0 \rightarrow F \rightarrow E[1] \rightarrow G \rightarrow 0$ in $\mathcal{A}^{\alpha_0, \beta_0}(X)$. By the definition of $\mathcal{A}^{\alpha_0, \beta_0}(X)$ we must have $\nu_{\alpha_0, \beta_0}(F) = \nu_{\alpha_0, \beta_0}(G) = 0$. The long exact sequence with respect to $\text{Coh}^{\beta_0}(X)$ leads to

$$0 \rightarrow \mathcal{H}_{\beta_0}^{-1}(F) \rightarrow E \rightarrow \mathcal{H}_{\beta_0}^{-1}(G) = G[-1] \rightarrow \mathcal{H}_{\beta_0}^0(F) \rightarrow 0.$$

Due to Lemma 2.2.22, the object $\mathcal{H}_{\beta_0}^0(F)$ is supported in dimension 0. Since E is ν_{α_0, β_0} -stable and $G \neq 0$, we must have $\mathcal{H}_{\beta_0}^{-1}(F) = 0$. Therefore, F is a sheaf supported in dimension 0. But that is a contradiction to the fact that we have an exact sequence $0 \rightarrow F[-1] \rightarrow E \rightarrow G[-1] \rightarrow 0$ in $\mathcal{A}^{\alpha, \beta}(X)$ for $(\alpha, \beta) \in U \cap P_V$ unless $F = 0$. Therefore, $E = G[-1]$ is stable. \square

At the hyperbola the Chern character of stable objects usually changes between v and $-v$. This comes hand in hand with objects leaving the heart while a shift of the object enters the heart. The next lemma deals with the question which shift is at which point in the category.

Lemma 4.1.3. *Let v be the Chern character of an object in $D^b(X)$, $\alpha_0 > 0$, $\beta_0 \in \mathbb{R}$ and $s > 0$ such that $\nu_{\alpha_0, \beta_0}(v) = 0$ and $H^2 v_1^\beta > 0$. Assume there is a path $\gamma : [0, 1] \rightarrow \overline{P_v}$ with $\gamma(1) = (\alpha_0, \beta_0)$, $\gamma([0, 1]) \subset P_v$, $E \in \mathcal{A}^{\gamma(t)}(X)$ is $\lambda_{\gamma(t), s}$ -semistable for all $t \in [0, 1]$ and $\text{ch}(E) = \pm v$. Then $E[1] \in \mathcal{A}^{\alpha_0, \beta_0}(X)$.*

Proof. The map $[0, 1] \rightarrow \mathbb{R}$, $t \mapsto \phi_{\gamma(t), s}(E)$ is continuous. Thus, there is $m \in \{0, 1\}$ such that $E[m] \in \mathcal{A}^{\alpha_0, \beta_0}(X)$ is $\lambda_{\alpha_0, \beta_0, s}$ -semistable. Assume $m = 0$. Then Lemma 2.2.22 implies that $\mathcal{H}_{\beta_0}^{-1}(E)$ is ν_{α_0, β_0} -semistable and $\mathcal{H}_{\beta_0}^0(E)$ is a sheaf supported in dimension 0. This implies $H^2 \text{ch}_1^{\beta_0}(E) \leq 0$. Therefore, $H^2 v_1^{\beta_0} > 0$ implies $\text{ch}(E) = -v$. This leads to

$$\mathfrak{S}Z_{\gamma(t), s}(E) = -\mathfrak{S}Z_{\gamma(t), s}(v) < 0$$

for all $t \in [0, 1)$ in contradiction to $E \in \mathcal{A}^{\alpha_0, \beta_0}(X)$. \square

The final lemma restricts the possibilities for semistable objects that leave the heart while a shift enters the heart.

Lemma 4.1.4. *Let $\gamma : [0, 1] \rightarrow \mathbb{R}_{>0} \times \mathbb{R}$ be a path, $\gamma(1) = (\alpha_0, \beta_0)$, $s > 0$, $E \in D^b(X)$ be an object such that $E \in \mathcal{A}^{\gamma(t)}(X)$ is $\lambda_{\gamma(t), s}$ -semistable for all $t \in [0, 1)$ and $E[1] \in \mathcal{A}^{\alpha_0, \beta_0}(X)$ is $\lambda_{\alpha_0, \beta_0, s}$ -semistable. Then $E \in \text{Coh}^{\beta_0}(X)$ is ν_{α_0, β_0} -semistable.*

Proof. The continuity of $[0, 1] \rightarrow \mathbb{R}$, $t \mapsto \phi_{\gamma(t), s}(E)$ implies $\mathfrak{S}Z_{\alpha_0, \beta_0, s}(E) = 0$. Then Lemma 2.2.22 implies that $\mathcal{H}_{\beta_0}^{-1}(E[1])$ is ν_{α_0, β_0} -semistable and $\mathcal{H}_{\beta_0}^0(E[1])$ is a sheaf supported in dimension 0. In particular, there is a non trivial map $E[1] \rightarrow \mathcal{H}_{\beta_0}^0(E[1])$ unless $\mathcal{H}_{\beta_0}^0(E[1]) = 0$. Since $E \in \mathcal{A}^{\gamma(t)}(X)$ for $t \in [0, 1)$ one obtains

$$\phi_{\gamma(t), s}(E[1]) > 1 = \phi_{\gamma(t), s}(\mathcal{H}_{\beta_0}^0(E[1])).$$

The semi-stability of E implies $\mathcal{H}_{\beta_0}^0(E[1]) = 0$. \square

Together with these three lemmas, we can prove the Theorem.

Proof of Theorem 4.1.1. We start by proving (i). Since $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ also defines a wall in $U \cap P_v$, we know there is $m \in \mathbb{Z}$ such that $E[m], F[m], G[m] \in \mathcal{A}^{\alpha, \beta}(X)$ for $(\alpha, \beta) \in U \cap P_v$. By Lemma 4.1.3 this implies $m = -1$ and Lemma 4.1.4 shows $E[-1], F[-1]$ and $G[-1]$ are all ν_{α_0, β_0} -semistable.

This defines a wall in tilt stability unless $\nu_{\alpha, \beta}(F) = \nu_{\alpha, \beta}(G)$ for all $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. But this is only possible if $\lambda_{\alpha, \beta, s}(F) = \lambda_{\alpha, \beta, s}(G)$ is equivalent to $\nu_{\alpha, \beta}(v) = 0$.

We continue by showing part (ii). By assumption (α_0, β_0) does not lie on any wall for v in tilt stability. Let U' be a neighborhood of (α_0, β_0) that does not intersect any such wall. In particular, this means $M_{\alpha, \beta}^{\text{tilt}}(v)$ is constant on U' . By part (i) any wall in Bridgeland stability that intersects the hyperbola $\{\nu_{\alpha, \beta}(v) = 0\}$ and stays an actual wall in some part of P_v comes from a wall in tilt stability. Therefore, we can choose a neighborhood U'' of (α_0, β_0) such that there is no wall in Bridgeland stability for v in $U'' \cap P_v$. We define $U := U' \cap U''$ and choose $(\alpha, \beta) \in U$.

The inclusion $M_{\alpha, \beta}^{\text{tilt}}(v) \subset M_{\alpha, \beta, s}(v)$ is a restatement of Lemma 4.1.2. Let $E \in M_{\alpha, \beta, s}(v)$. There is $m \in \mathbb{Z}$ such that $E[m] \in \mathcal{A}^{\alpha_0, \beta_0}$ is a $\lambda_{\alpha_0, \beta_0, s}$ -semistable object. By Lemma 4.1.3 one gets $m = 1$ and Lemma 4.1.4 implies $E \in \text{Coh}^\beta(X)$ is tilt semistable, i.e. $E \in M_{\alpha, \beta}^{\text{tilt}}(v)$.

Part (iii) follows from (ii) while (iv) is an immediate application of Lemma 4.1.2.

□

4.2 Line Bundles on Projective Space

In the case of \mathbb{P}^3 more can be proven than in the general case. It was already shown in [BMT14] that a line bundle L is tilt stable if $Q^{\text{tilt}}(L) = 0$. This condition

always holds in Picard rank 1. However, we need a slightly more refined result that holds in the special case of \mathbb{P}^3 .

Proposition 4.2.1. *Let $v = \pm \text{ch}(\mathcal{O}(n)^{\oplus m})$ for integers n, m with $m > 0$. Then $\mathcal{O}(n)^{\oplus m}$ or a shift of it is the unique tilt semistable and Bridgeland semistable object with Chern character $\pm v$ for any $\alpha > 0$ and β . Moreover, in the case $m = 1$ the line bundle $\mathcal{O}(n)$ is stable.*

For the proof we will need a connection between Bridgeland stability and quiver representations. In the case of the quadric threefold we did this in the previous chapter. In the case of \mathbb{P}^3 a similar argument was carried out in [Mac14b].

Theorem 4.2.2. *If $\alpha < 1/3$ and $\beta \in (-2/3, 0]$ then*

$$\mathcal{C} := \langle \mathcal{O}(-1)[3], T(-2)[2], \mathcal{O}[1], \mathcal{O}(1) \rangle = P_{\alpha, \beta, s}((\phi, \phi + 1))$$

for some $\phi \in (0, 1)$ and the Bridgeland stability condition $(P_{\alpha, \beta, s}, Z_{\alpha, \beta, s})$ for small enough $s > 0$. Moreover, \mathcal{C} is the category $\text{mod } -A$ for some finite dimensional algebra A coming from an exceptional collection as in Theorem 3.3.3. The four objects generating \mathcal{C} correspond to the simple representations.

Proof of Proposition 4.2.1. By using the autoequivalence given by tensoring with $\mathcal{O}(-n)$, we can reduce to the case $n = 0$. Then $v = \pm(m, 0, 0)$.

We start by proving the statement in Bridgeland stability for $\alpha = \frac{1}{4}$ and $\beta = 0$. By Theorem 4.2.2 the object $\mathcal{O}[1]$ corresponds to a simple representation at this point. Then any object E in the quiver category with $\text{ch}(E) = v$ corresponds to a representation of the form $0 \rightarrow 0 \rightarrow \mathbb{C}^m \rightarrow 0$. The statement follows in this case, since there is a unique such representation and it is semistable.

Next, we will extend this to all α, β . Notice that $Q_{\alpha,\beta,K}(v) = 0$. By Lemma 2.2.5 the object \mathcal{O} is Bridgeland stable for all α, β . Let $E \in \mathcal{A}^{\alpha,\beta}(\mathbb{P}^3)$ be $Z_{\alpha,\beta,s}$ -semistable with $\text{ch}(E) = v$. By Lemma 2.2.5, the class v spans an extremal ray of the cone $\mathcal{C}^+ = Z_{\alpha,\beta,s}^{-1}(\mathbb{R}_{\geq 0}v) \cap \{Q_{\alpha,\beta,K} \geq 0\}$. In particular, that means all its Jordan-Hölder factors are scalar multiples of v . If $m = 1$, then v is primitive in the lattice. Therefore, E is actually stable and then E is also stable for $\alpha = \frac{1}{4}$ and $\beta = 0$, i.e. E is \mathcal{O} or a shift of it. Assume $m > 1$. Since there are no stable objects with class v at $\alpha = \frac{1}{4}$ and $\beta = 0$, Lemma 2.2.5 implies that E is strictly semistable. Therefore, the case $m = 1$ implies that all the Jordan-Hölder factors are \mathcal{O} .

The next step is to show semistability of \mathcal{O}^m in tilt stability. For this, we just need deal with $m = 1$. We have $Q^{\text{tilt}}(\mathcal{O}) = 0$. By Lemma 2.2.5 we know that \mathcal{O} is tilt stable everywhere or nowhere unless it is destabilized by an object supported in dimension 0. In that case $\beta = 0$ is a wall. However, that cannot happen since there are no morphism from or to $\mathcal{O}[1]$ for any skyscraper sheaf. Since v is primitive, semistability of \mathcal{O} is equivalent to stability. For $\beta = 0$ and $\alpha \gg 0$ we know that \mathcal{O} is semistable due to Lemma 2.2.13.

Now we will show that any tilt semistable object E with $\text{ch}(E) = v$ has to be \mathcal{O}^m for $\alpha = 1, \beta = -1$. We have $\nu_{1,-1}(E) = 0$. Therefore, $E[1]$ is in the category $\mathcal{A}^{1,-1}(\mathbb{P}^3)$. The Bridgeland slope is $\lambda_{1,-1,s}(E[1]) = \infty$ independently of s . This means E is Bridgeland semistable and by the previous argument $E \cong \mathcal{O}^m$.

We will use $Q^{\text{tilt}}(v) = 0$ and Lemma 2.2.5 similarly as in the Bridgeland stability case to extend it to all of tilt stability. We start with the case $\beta < 0$. Let $E \in \text{Coh}^\beta(\mathbb{P}^3)$ be a tilt semistable object with $\text{ch}(E) = v$. By using Lemma 2.2.5, the class v spans an extremal ray of the cone $\mathcal{C}^+ = (Z_{\alpha,\beta}^{\text{tilt}})^{-1}(\mathbb{R}_{\geq 0}v) \cap \{Q^{\text{tilt}} \geq 0\}$. In

particular, that means all its stable factors have Chern character $(1, 0, 0, e)$. The BMT inequality shows $e \leq 0$. But since all the stable factor add up to v this means $e = 0$. Therefore, we reduced to the case $m = 1$. In this case Lemma 2.2.5 does the job as before.

If $\beta = 0$, the situation is more involved, since skyscraper sheaves can be stable factors. All stable factor have Chern characters of the form $(-1, 0, 0, e)$ or $(0, 0, 0, f)$. In this case $f \geq 0$. Let F be such a stable factor with Chern character $(-1, 0, 0, e)$. By openness of stability F is stable in a whole neighborhood that includes points with $\beta < 0$ and $\beta > 0$. The BMT-inequality in both cases together implies $e = 0$. But then $f = 0$ follows from the fact that Chern characters are additive. Again we reduced to the case $m = 1$. By openness of stability and the result for $\beta < 0$ we are done with this case. The case $\beta > 0$ can now be handled in the same way as $\beta < 0$ by using Lemma 2.2.5 again. \square

In the case of tilt stability there is an even stronger statement. If $\beta > n$, we do not need to fix ch_3 to get the same conclusion.

Proposition 4.2.3. *Let $v = -\text{ch}_{\leq 2}(\mathcal{O}(n)^{\oplus m})$ for integers n, m with $m > 0$. Then $\mathcal{O}(n)^{\oplus m}[1]$ is the unique tilt semistable object E with $\text{ch}_{\leq 2}(E) = v$ for any $\alpha > 0$ and $\beta > n$.*

Proof. The semistability of $\mathcal{O}(n)^{\oplus m}[1]$ has already been shown in Proposition 4.2.1. As in the previous proof, we can use tensoring by $\mathcal{O}(-n)$ to reduce to the case $n = 0$. This means $v = (m, 0, 0)$.

Let $E \in \text{Coh}^{\beta}(\mathbb{P}^3)$ be a tilt stable object for some $\alpha > 0$ and $\beta > 0$ with $\text{ch}(E) = (-m, 0, 0, e)$. The BMT-inequality implies $e \leq 0$. Since $Q^{\text{tilt}}(E) = 0$, we can use

Lemma 2.2.5 to get that E is tilt stable for all $\beta > 0$. If E is also stable for $\beta = 0$, then using the BMT-inequality for $\beta < 0$ implies $e = 0$. Assume E becomes strictly semistable at $\beta = 0$. By Lemma 2.2.5 the class v spans an extremal ray of the cone $\mathcal{C}^+ = (Z_{\alpha,\beta}^{\text{tilt}})^{-1}(\mathbb{R}_{\geq 0}v) \cap \{Q^{\text{tilt}} \geq 0\}$. That means all stable factors must have Chern characters of the form $(-m', 0, 0, e')$ for some $0 \leq m' \leq m$. If $m' \neq 0$ then using the BMT-inequality for both $\beta < 0$ and $\beta > 0$ implies $e' = 0$. If $m' = 0$, then $e' > 0$. However, all the third Chern characters add up to the non positive number e . This is only possible if $e = 0$ and no stable factor has $m' = 0$. By Proposition 4.2.1 this means $E \cong \mathcal{O}[1]^m$ and since E is stable this is only possible if $m = 1$.

Let $E \in \text{Coh}^\beta(\mathbb{P}^3)$ be a strictly tilt semistable object for some $\alpha > 0$ and $\beta > 0$ with $\text{ch}_{\leq 2}(E) = (-m, 0, 0)$. Since $Q^{\text{tilt}}(E) = 0$, we can use Lemma 2.2.5 again to get that all stable factors F have $\text{ch}_{\leq 2}(F) = (-m', 0, 0)$ for some $m' > 0$. By the previous part of the proof this means $m' = 1$ and $F \cong \mathcal{O}[1]$ finishes the proof. \square

We finish this section by recalling a basic characterization of ideal sheaves in \mathbb{P}^k .

Lemma 4.2.4. *Let $E \in \text{Coh}(\mathbb{P}^k)$ be torsion free of rank one and $\text{ch}_1(E) = 0$. Then either $E \cong \mathcal{O}$ or there is a subscheme $Z \subset \mathbb{P}^k$ of codimension at least two such that $E \cong \mathcal{I}_Z$.*

Proof. We have the inclusion $E \hookrightarrow E^{\vee\vee}$. The sheaf $E^{\vee\vee}$ is reflexive of rank one, i.e. locally free (see [Har80, Chapter 1] for basic properties of reflexive sheaves). Due to $\text{ch}_1(E) = 0$ and $\text{rk}(E) = 1$, we get $E^{\vee\vee} \cong \mathcal{O}$. Therefore, either $E \cong \mathcal{O}$ or there is a subscheme $Z \subset \mathbb{P}^k$ such that $E \cong \mathcal{I}_Z$. If Z is not of codimension at least two, then $c_1(E) \neq 0$. \square

4.3 Some Sheaves on Projective Space

4.3.1 Tilt Stability

Let $m, n \in \mathbb{Z}$ be integers with $n < m$ and $i, j \in \mathbb{N}$ positive integers. We define a class as $v = i \operatorname{ch}(\mathcal{O}_{\mathbb{P}^3}(m)) - j \operatorname{ch}(\mathcal{O}_{\mathbb{P}^3}(n))$. In this section we study walls for this class v in tilt stability. Interesting examples of sheaves with this Chern character include ideal sheaves of complete intersections of two surfaces of the same degree or ideal sheaves of twisted cubics. In this generality we will determine the smallest wall in tilt stability on one side of the vertical wall.

Theorem 4.3.1. *A wall not containing any smaller wall for $M_{\alpha,\beta}^{\text{tilt}}(v)$ is given by the equation $\alpha^2 + (\beta - \frac{m+n}{2})^2 = (\frac{m-n}{2})^2$. All semistable objects E at the wall are given by extensions of the form $0 \rightarrow \mathcal{O}(m)^{\oplus i} \rightarrow E \rightarrow \mathcal{O}(n)^{\oplus j}[1] \rightarrow 0$. Moreover, there are no tilt semistable objects inside this semicircle.*

Proof. The semicircle defined by $Q_{\alpha,\beta}(v) = 0$ coincides with the wall claimed to exist. Therefore, the BMT-inequality implies that no smaller semicircle can be a wall. Moreover, Proposition 4.2.1 shows that both $\mathcal{O}(m)^{\oplus i}$ and $\mathcal{O}(n)^{\oplus j}[1]$ are tilt semistable. The equation $\nu_{\alpha,\beta}(\mathcal{O}(m)) = \nu_{\alpha,\beta}(\mathcal{O}(n))$ is equivalent to $\alpha^2 + (\beta - \frac{m+n}{2})^2 = (\frac{m-n}{2})^2$. Therefore, we are left to prove the second assertion.

Let F be a stable factor of E at the wall. By Lemma 2.2.7 and Remark 2.2.8 we get $Q_{\alpha,\beta}(F) = 0$ at the wall. Since F is stable, it is stable in a whole neighborhood around the wall. But $Q_{\alpha,\beta}(F)$ will be negative on one side of the wall unless $Q_{\alpha,\beta}(F) = 0$ for all α, β . This implies $Q^{\text{tilt}}(F) = 0$.

Assume that $\operatorname{ch}(F) = (r, c, d, e)$. Then $Q^{\text{tilt}}(F) = 0$ implies $c^2 - 2rd = 0$. If $r = 0$, then $c = 0$. That cannot happen since the wall would be a vertical line and not a

semicircle in that situation. Thus, we can assume $r \neq 0$. In particular, the equality $d = \frac{c^2}{2r}$ holds. The equation $Q_{\alpha,\beta}(F) = 0$ for all (α, β) implies $e = \frac{c^3}{6r^2}$. In particular, the point $\alpha_0 = \frac{m-n}{2}$, $\beta_0 = \frac{m+n}{2}$ lies on the wall. Since F and E have the same slope at (α_0, β_0) , a straightforward but lengthy computation shows $c = mr$ or $c = nr$. That means $\text{ch}(F)$ is a multiple of the Chern character of either $\mathcal{O}(m)$ or $\mathcal{O}(n)$. Since F was assumed to be stable, Proposition 4.2.1 shows that F has to be one of those line bundles.

Since the Chern characters of these two line bundles are linearly independent we know that any decomposition of E into stable factors must contain i times $\mathcal{O}(m)$ and j times $\mathcal{O}(n)[1]$. The proof is finished by $\text{Ext}^1(\mathcal{O}(m), \mathcal{O}(n)[1]) = 0$. \square

In the case of the Chern character of an ideal sheaf of a curve there is also a bound on the biggest wall.

Proposition 4.3.2. *Let $v = (1, 0, -d, e)$ be the Chern character of an ideal sheaf of a curve of degree d . The biggest wall for $M_{\alpha,\beta}^{\text{tilt}}(v)$ and $\beta < 0$ is contained inside the semicircle defined by $\nu_{\alpha,\beta}(v) = \nu_{\alpha,\beta}(\mathcal{O}(-1))$. The biggest wall in the case $\beta > 0$ is contained inside the semicircle defined by $\nu_{\alpha,\beta}(v) = \nu_{\alpha,\beta}(\mathcal{O}(1))$.*

Proof. We have to show there is no wall intersecting $\beta = \pm 1$. Let E be tilt semistable for $\beta = \pm 1$ and some α with $\text{ch}(E) = \pm v$. Then $\text{ch}_1^{\pm 1}(E) = 1$ holds. If E is strictly tilt semistable, then there is an exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ of tilt semistable objects with the same slope. However, either $\text{ch}^{\pm 1}(F) = 0$ or $\text{ch}^{\pm 1}(G) = 0$, a contradiction. The numerical wall $\nu_{\alpha,\beta}(v) = \nu_{\alpha,\beta}(\mathcal{O}(\pm 1))$ contains the point $\alpha = 0$, $\beta = \pm 1$. The argument is finished by the fact that numerical walls cannot intersect. \square

4.3.2 Bridgeland Stability

We will show that there is a path close to one branch of the hyperbola defined by $\Im Z_{\alpha,\beta,s}(v) = 0$ where the last wall crossing described in Theorem 4.3.1 happens in Bridgeland stability. The first moduli space after this wall turns out to be smooth and irreducible. Moreover, at the beginning of the path stable objects are exactly slope stable sheaves with Chern character v .

Theorem 4.3.3. *Assume (v_0, v_1, v_2) is a primitive vector. There is a path $\gamma : [0, 1] \rightarrow \mathbb{R}_{>0} \times \mathbb{R} \subset \text{Stab}(\mathbb{P}^3)$ that satisfies the following properties.*

(i) *The first wall on γ is given by $\lambda_{\alpha,\beta,s}(\mathcal{O}(m)) = \lambda_{\alpha,\beta,s}(\mathcal{O}(n))$. Before the wall there are no semistable objects. After the wall the moduli space is smooth, irreducible and projective.*

(ii) *At $\gamma(1)$ the semistable objects are exactly slope stable coherent sheaves E with $\text{ch}(E) = v$. Moreover, there are no strictly semistable objects.*

Proof. By Theorem 4.3.1 there is a wall in tilt stability defined by the equation $\nu_{\alpha,\beta}(\mathcal{O}(m)) = \nu_{\alpha,\beta}(\mathcal{O}(n))$. Moreover, there is no smaller wall. Since (v_0, v_1, v_2) is a primitive vector, any moduli space of $\nu_{\alpha,\beta}$ -semistable objects for v , such that (α, β) does not lie on a wall, consists solely of tilt stable objects. Let $Y \subset \{\Im Z_{\alpha,\beta,s}(v) = 0\}$ be the branch of the hyperbola that intersects this wall. Due to Theorem 4.1.1 we can find a path $\gamma : [0, 1] \rightarrow \mathbb{R}_{>0} \times \mathbb{R} \hookrightarrow \text{Stab}(\mathbb{P}^3)$ close enough to Y such that all moduli spaces of tilt stable objects that occur on Y outside of any wall are moduli spaces of Bridgeland stable objects along γ . Moreover, we can assume that γ intersects no wall twice and the first wall crossing is given by $\lambda_{\alpha,\beta,s}(\mathcal{O}(m)) = \lambda_{\alpha,\beta,s}(\mathcal{O}(n))$.

Part (ii) can be proven as follows. By the choice of γ , we have $M_{\gamma(1)}^{\text{tilt}}(v) = M_{\gamma(1),s}(v)$. In tilt stability $\gamma(1)$ is above the largest wall. Therefore, Lemma 2.2.13 and Lemma 2.2.14 imply that $M_{\gamma(1)}^{\text{tilt}}(v)$ consists of slope stable sheaves E with $\text{ch}(E) = v$.

We will finish the proof of (i) by showing that the first moduli space is a moduli space of representations on a Kronecker quiver. Let $t \in (0, 1)$ be such that $M_{\gamma(t),s}(v)$ is the first moduli space on γ after the empty space. Let Q be the Kronecker quiver with $N = \dim \text{Hom}(\mathcal{O}(n), \mathcal{O}(m))$ arrows.

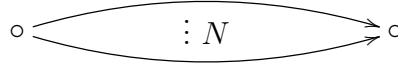


Figure 4.1: Kronecker quiver

Since we know that the first moduli space consists solely of extensions of $\mathcal{O}(n)^{\oplus j}[1]$ and $\mathcal{O}(m)^{\oplus i}$, we can find θ such that θ -stability and Bridgeland stability at $\gamma(t)$ match. More precisely, there is a bijection between Bridgeland stable objects at $\gamma(t)$ with Chern character v and θ -stable complex representations with dimension vector (j, i) . We denote this specific moduli space of quiver representations by K . Since the quiver has no relation and i, j have to be coprime, we get that K is a smooth projective variety.

We want to construct an isomorphism between K and the moduli space $M_{\gamma(t),s}(v)$ of Bridgeland stable complexes with Chern character v . In order to do so, we need to make the above bijection more precise. Let $\text{Hom}(\mathcal{O}(n), \mathcal{O}(m)) = \bigoplus_l \mathbb{C}\varphi_l$. There is a functor $\mathcal{F} : \text{Rep}(Q) \rightarrow D^b(\mathbb{P}^3)$ that sends a representation $f_l : \mathbb{C}^j \rightarrow \mathbb{C}^i$ to the two

term complex $\mathcal{O}(n)^{\oplus j} \rightarrow \mathcal{O}(m)^{\oplus i}$ with map $(s_1, \dots, s_j) \mapsto \sum_l f_l(\varphi_l(s_1), \dots, \varphi_l(s_j))$. This functor induces the bijection between stable objects mentioned above.

The functor above can be generalized to the relative setting as $\mathcal{F}_S : \text{Rep}_S(Q) \rightarrow D^b(\mathbb{P}^3 \times S)$ sending $f_l : V \rightarrow W$ to the two term complex $V \boxtimes \mathcal{O}(n) \rightarrow W \boxtimes \mathcal{O}(m)$ where the map is given by $\sum v \otimes s \mapsto \sum \sum_l f_l(v) \otimes \varphi_l(s)$.

If \mathcal{E} is a family of Bridgeland stable objects at $\gamma(t)$ over S , then we get $\mathcal{F}(\mathcal{E}_s) = \mathcal{F}_S(\mathcal{E})_s$ for any $s \in S$. That induces a bijective morphism from K to $M_{\gamma(t),s}(v)$. We want to show that this morphism is in fact an isomorphism. In order to so, we will first need to prove smoothness.

We have $\dim M_{\gamma(t),s}(v) = \dim K = jiN - i^2 - j^2 + 1$. For any $E \in M_{\gamma(t),s}(v)$ the Zariski tangent space at E is given by $\text{Ext}^1(E, E)$. We have an exact triangle

$$O(m)^{\oplus i} \rightarrow E \rightarrow O(n)^{\oplus j}[1]. \quad (4.1)$$

Since E is stable we have $\text{Hom}(O(n)[1], E) = 0$. Applying $\text{Hom}(O(n), \cdot)$ to (4.1) leads to $\text{Hom}(O(n), E) = \mathbb{C}^{Ni-j}$. The same way we get $\text{Hom}(O(m), E) = \mathbb{C}^i$ and $\text{Ext}^1(O(m), E) = 0$. Since E is stable, the equation $\text{Hom}(E, E) = \mathbb{C}$ holds. Applying $\text{Hom}(\cdot, E)$ to (4.1) leads to the following long exact sequence.

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^{i^2} \rightarrow \mathbb{C}^{Nij-j^2} \rightarrow \text{Ext}^1(E, E) \rightarrow 0.$$

That means $\dim \text{Ext}^1(E, E) = Nij - j^2 - i^2 + 1 = \dim M_{\gamma(t),s}(v)$, i.e. $M_{\gamma(t),s}(v)$ is smooth.

Since there are no strictly semistable objects, we can use Theorem 2.3.7 to infer that $M_{\gamma(t),s}(v)$ is a smooth proper algebraic space of finite type over \mathbb{C} . According to [Knu71, Page 23] there is a fully faithful functor from smooth proper algebraic

spaces of finite type over \mathbb{C} to complex manifolds. Since any bijective holomorphic map between two complex manifolds has a holomorphic inverse we are done. \square

Chapter 5: Concrete Examples

In this chapter, we compute the examples of twisted cubics and elliptic quartics. The content is mostly based on [Sch15] and [GHS16].

In general the BMT-inequality is not sharp. We believe that the correct approach for computing walls in tilt stability is based on an induction on the positive integer discriminant Q^{tilt} . A first step for low values of Q^{tilt} is proven in the next lemma.

Lemma 5.0.1. *Let $\beta \in \mathbb{Z}$ and $E \in \text{Coh}^\beta(\mathbb{P}^3)$ be tilt semistable.*

(i) *If $\text{ch}^\beta(E) = (1, 1, d, e)$ then $d - 1/2 \in \mathbb{Z}_{\leq 0}$. In the case $d = -1/2$, we get $E \cong \mathcal{I}_L(\beta + 1)$ where L is a line plus $1/6 - e$ (possibly embedded) points in \mathbb{P}^3 . If $d = 1/2$, then $E \cong \mathcal{I}_Z(\beta + 1)$ for a zero dimensional subscheme $Z \subset \mathbb{P}^3$ of length $1/6 - e$.*

(ii) *If $\text{ch}^\beta(E) = (0, 1, d, e)$, then $d + 1/2 \in \mathbb{Z}$ and $E \cong I_{Z/V}(\beta + d + 1/2)$ where Z is a dimension zero subscheme of length $1/24 + d^2/2 - e$.*

Proof. Lemma 2.2.13 implies E to be either a torsion free sheaf or a pure sheaf supported in dimension 2. By tensoring E with $\mathcal{O}(-\beta)$ we can reduce to the case $\beta = 0$.

In case (i) we have $\text{ch}(E \otimes \mathcal{O}(-1)) = (1, 0, d - 1/2, 1/3 - d + e)$. Lemma 4.2.4 implies that $E \otimes \mathcal{O}(-1)$ is an ideal sheaf of a subscheme $Z \subset \mathbb{P}^3$. This implies

$d - 1/2 \in \mathbb{Z}_{\leq 0}$. If $d = 1/2$, then Z is zero dimensional of length $d - e - 1/3 = 1/6 - e$. In case $d = -1/2$, the subscheme Z is a line plus points. The Chern Character of the ideal sheaf of a line is given by $(1, 0, -1, 1)$. Therefore, the number of points is $1 + d - e - 1/3 = 1/6 - e$.

In case (ii) E is supported on a plane V . We will use Lemma 4.2.4 on V . In order to so, we need to use the Grothendieck-Riemann-Roch Theorem to compute the Chern character of E on V . The Todd classes of \mathbb{P}^2 and \mathbb{P}^3 are given by $\text{td}(\mathbb{P}^2) = (1, \frac{3}{2}, 1)$ and $\text{td}(\mathbb{P}^3) = (1, 2, \frac{11}{6}, 1)$. Therefore, we get

$$\begin{aligned} i_* \left(\text{ch}_V(E) \left(1, \frac{3}{2}, 1 \right) \right) &= (0, 1, d, e) \left(1, 2, \frac{11}{6}, 1 \right) \\ &= \left(0, 1, d + 2, 2d + e + \frac{11}{6} \right) \end{aligned}$$

where $i : V \hookrightarrow \mathbb{P}^3$ is the inclusion. Thus, we have $\text{ch}_V(E) = (1, d + 1/2, d/2 + e + 1/12)$ and $d + 1/2$ is indeed an integer. Moreover, we can compute

$$\text{ch}_V(E \otimes \mathcal{O}(-d - 1/2)) = \left(1, 0, e - \frac{d^2}{2} - \frac{1}{24} \right).$$

Using Lemma 4.2.4 on V concludes the proof. □

5.1 Twisted Cubics

In this section we will compute walls in the case of twisted cubic curves C in \mathbb{P}^3 .

5.1.1 Tilt Stability

The locally free resolution $0 \rightarrow \mathcal{O}(-3)^{\oplus 2} \rightarrow \mathcal{O}(-2)^{\oplus 3} \rightarrow \mathcal{I}_C \rightarrow 0$ implies

$$\text{ch}^\beta(\mathcal{I}_C) = \left(1, -\beta, \frac{\beta^2}{2} - 3, -\frac{\beta^3}{6} + 3\beta + 5 \right).$$

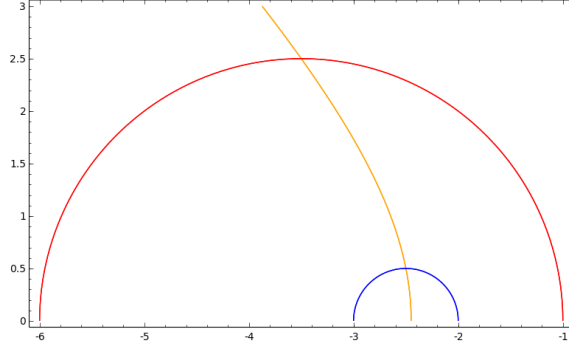


Figure 5.1: Walls in tilt stability for twisted cubics

Theorem 5.1.1. *There are two walls for $M_{\alpha,\beta}^{\text{tilt}}(1, 0, -3, 5)$ for $\alpha > 0$ and $\beta < 0$. Moreover, the following table lists pairs of tilt semistable objects whose extensions completely describe all strictly semistable objects at each of the corresponding walls. Let V be a plane in \mathbb{P}^3 , $P \in \mathbb{P}^3$ and $Q \in V$.*

$\alpha^2 + (\beta + \frac{5}{2})^2 = (\frac{1}{2})^2$	$\mathcal{O}(-2)^{\oplus 3}, \mathcal{O}(-3)[1]^{\oplus 2}$
$\alpha^2 + (\beta + \frac{7}{2})^2 = (\frac{5}{2})^2$	$\mathcal{I}_P(-1), \mathcal{O}_V(-3)$ $\mathcal{O}(-1), \mathcal{I}_{Q/V}(-3)$

Table 5.1: Objects defining walls for twisted cubics

The hyperbola $\nu_{\alpha,\beta}(1, 0, -3) = 0$ is given by the equation $\beta^2 - \alpha^2 = 6$.

In order to prove the Theorem, we need the following lemma that determines the Chern characters of possibly destabilizing objects for $\beta = -2$.

Lemma 5.1.2. *If an exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ in $\text{Coh}^{-2}(\mathbb{P}^3)$ defines a wall for $\beta = -2$ with $\text{ch}_{\leq 2}(E) = (1, 0, -3)$ then up to interchanging F and G we have $\text{ch}_{\leq 2}^{-2}(F) = (1, 1, \frac{1}{2})$ and $\text{ch}_{\leq 2}^{-2}(G) = (0, 1, -\frac{3}{2})$.*

Proof. The argument is completely independent of F being a quotient or a subobject.

We have $\text{ch}_{\leq 2}^{-2}(E) = (1, 2, -1)$.

Let $\text{ch}_{\leq 2}^{-2}(F) = (r, c, d)$. By definition of $\text{Coh}^{-2}(\mathbb{P}^3)$, we have $0 \leq c \leq 2$. If $c = 0$, then $\nu_{\alpha, -2}(F) = \infty$ and this is in fact no wall for any $\alpha > 0$. If $c = 2$, then the same argument for the quotient G shows there is no wall. Therefore, $c = 1$ must hold. We can compute

$$\nu_{\alpha, -2}(E) = -\frac{2 + \alpha^2}{4}, \quad \nu_{\alpha, -2}(F) = d - \frac{r\alpha^2}{2}.$$

The wall is defined by $\nu_{\alpha, -2}(E) = \nu_{\alpha, -2}(F)$. This leads to

$$\alpha^2 = \frac{4d + 2}{2r - 1} > 0. \tag{5.1}$$

The next step is to rule out the cases $r \geq 2$ and $r \leq -1$. If $r \geq 2$, then $\text{rk}(G) \leq -1$. By exchanging the roles of F and G in the following argument, it is enough to deal with the situation $r \leq -1$. In that case we use (5.1) and the Bogomolov inequality to get the contradiction $2rd \leq 1$, $d < -\frac{1}{2}$ and $r \leq -1$.

Therefore, we know $r = 0$ or $r = 1$. By interchanging the roles of F and G if necessary we only have to handle the case $r = 1$. Equation (5.1) implies $d > -\frac{1}{2}$. By Lemma 5.0.1 we get $d - 1/2 \in \mathbb{Z}_{\leq 0}$. Therefore, we are left with the case in the claim. □

Proof of Theorem 5.1.1. Since we are only dealing with $\beta < 0$ the structure theorem for walls in tilt stability implies that all walls intersect the left branch of the hyperbola.

In Theorem 4.3.1 we already determined the smallest wall in much more generality. This semicircle intersects the β -axis at $\beta = -3$ and $\beta = -2$. Therefore, all other walls intersecting this branch of the hyperbola have to intersect the ray $\beta = -2$. By Lemma 5.1.2 there is at most one wall on this ray. It corresponds to the solution claimed to exist.

Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ define a wall in $\text{Coh}^{-2}(\mathbb{P}^3)$ with $\text{ch}(E) = (1, 0, -3, 5)$. One can compute $\text{ch}^{-2}(E) = (1, 2, -1, \frac{1}{3})$. Up to interchanging the roles of F and G we have $\text{ch}^{-2}(F) = (1, 1, 1/2, e)$ and $\text{ch}^{-2}(G) = (0, 1, -3/2, 1/3 - e)$. By Lemma 5.0.1 we get $F \cong \mathcal{I}_Z(-1)$ where $Z \in \mathbb{P}^3$ is a zero dimensional sheaf of length $1/6 - e$ in \mathbb{P}^3 . In particular, the inequality $e \leq 1/6$ holds. The same lemma also implies that $G \cong I_{Z'/V}(-3)$ where Z' is a dimension zero subscheme of length $e + 5/6$ in V . In particular, $e \geq -5/6$. Therefore, the two cases $e = \frac{1}{6}$ and $e = -\frac{5}{6}$ remain and correspond exactly to the two sets of objects in the Theorem. \square

5.1.2 Bridgeland Stability

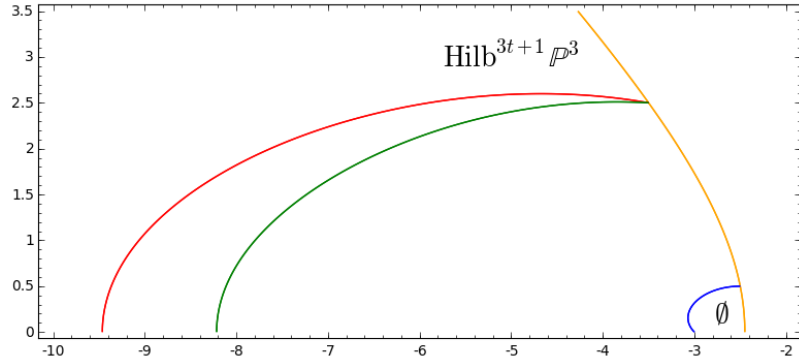


Figure 5.2: Walls in Bridgeland stability for twisted cubics

After describing all walls in tilt stability for $\beta < 0$ in Theorem 5.1.1, we will translate this result into Bridgeland stability via Theorem 4.1.1.

Theorem 5.1.3. *There is a path $\gamma : [0, 1] \rightarrow \mathbb{R}_{>0} \times \mathbb{R} \subset \text{Stab}(\mathbb{P}^3)$ that crosses the following walls for $v = (1, 0, -3, 5)$ in the following order. The walls are defined by the two given objects having the same slope. Moreover, all strictly semistable objects at each of the walls are extensions of those two objects. Let V be a plane in \mathbb{P}^3 , $P \in \mathbb{P}^3$ and $Q \in V$.*

(i) $\mathcal{O}(-2)^{\oplus 3}, \mathcal{O}(-3)[1]^{\oplus 2}$

(ii) $\mathcal{I}_P(-1), \mathcal{O}_V(-4)$

(iii) $\mathcal{O}(-1), \mathcal{I}_{Q/V}(-4)$

The chambers separated by those walls exhibit the following moduli spaces.

(i) The empty space $M_0 = \emptyset$.

(ii) A smooth projective variety M_1 .

(iii) A space with two components $M_2 \cup M'_2$. The space M_2 is a blow up of M_1 in the incidence variety parametrizing a point in a plane in \mathbb{P}^3 . The second component M'_2 is a \mathbb{P}^9 -bundle over the smooth variety $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$ parametrizing pairs $(\mathcal{I}_P(-1), \mathcal{O}_V(-4))$. The two components intersect transversally in the exceptional locus of the blow up.

(iv) The Hilbert scheme of curves C with $\text{ch}(\mathcal{I}_C) = (1, 0, -3, 5)$. It is given as $M_2 \cup M'_3$ where M'_3 is a blow up of M'_2 in the smooth locus parametrizing objects $\mathcal{I}_{Q/V}(-4)$.

Proof. Let γ be the path that exists due to Theorem 4.3.3. The fact that all the walls on this path occur in this form is a direct consequence of Theorem 4.1.1 and Theorem 5.1.1.

By Theorem 4.3.3 we know that $M_0 = \emptyset$, that M_1 is smooth, projective and irreducible and that the Hilbert scheme occurs at the end of the path. The main result in [PS85] is that this Hilbert scheme has exactly two smooth irreducible components of dimension 12 and 15 that intersect transversally in a locus of dimension 11. The 12-dimensional component M_2 contains the space of twisted cubics as an open subset. The 15-dimensional component M_3' parametrizes plane cubic curves with a potentially but no necessarily embedded point. Moreover, the intersection parametrizes plane singular cubic curves with a spatial embedded point at a singularity. In particular, those curves are not scheme theoretically contained in a plane.

Strictly semistable objects at the biggest wall are given by extensions of $\mathcal{O}(-1)$, $\mathcal{I}_{Q/V}(-4)$. For an ideal sheaf of a curve this can only mean that there is an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow I_C \rightarrow \mathcal{I}_{Q/V}(-4).$$

This can only exist if $C \subset V$ scheme theoretically. Therefore, the first wall does only modify the second component. The moduli space of objects $\mathcal{I}_{Q/V}(-4)$ is the incidence variety of points in the plane inside $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$. In particular, it is smooth and of dimension 5. A straightforward computation shows $\text{Ext}^1(\mathcal{O}(-1), \mathcal{I}_{Q/V}(-4)) = \mathbb{C}$. That means at the first wall the irreducible locus of extensions $\text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{O}(-1)) = \mathbb{C}^{10}$ is contracted onto a smooth locus. Moreover, for each sheaf $\mathcal{I}_{Q/V}(-4)$ the fiber is given by \mathbb{P}^9 . This means the contracted locus is a divisor. By a classical result of Moishezon [Moi67] any proper birational morphism $f : X \rightarrow Y$ between smooth

projective varieties such that the contracted locus E is irreducible and the image $f(E)$ is smooth is the blow up of Y in $f(E)$. Therefore, to see that M'_3 is the blow up of M'_2 we need to show that M'_2 is smooth.

At the second wall strictly semistable objects are given by extensions of $\mathcal{I}_P(-1)$ and $\mathcal{O}_V(-4)$. One computes the equalities $\text{Ext}^1(\mathcal{I}_P(-1), \mathcal{O}_V(-4)) = \mathbb{C}$ for $P \in V$, $\text{Ext}^1(\mathcal{I}_P(-1), \mathcal{O}_V(-4)) = 0$ for $P \notin V$ and $\text{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_P(-1)) = \mathbb{C}^{10}$. The objects $\mathcal{I}_P(-1)$ and $\mathcal{O}_V(-4)$ vary in \mathbb{P}^3 respectively $(\mathbb{P}^3)^\vee$ that are both fine moduli spaces. Therefore, the component M'_2 is a \mathbb{P}^9 -bundle over the moduli space of pairs $(\mathcal{O}_V(-4), \mathcal{I}_P(-1))$, i.e. $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$. This means M'_2 is smooth and projective.

We are left to show that M_2 is the blow up of M_1 . We already know that M_2 is the smooth component of the Hilbert scheme containing twisted cubic curves. Moreover, M_1 is smooth by Theorem 4.3.3. We want to apply the above result of Moishezon again. The exceptional locus of the map from M_2 to M_1 is given by the intersection of the two components in the Hilbert scheme. By [PS85] this is an irreducible divisor in M_2 . Due to $\text{Ext}^1(\mathcal{I}_P(-1), \mathcal{O}_V(-4)) = \mathbb{C}$ for $P \in V$ the image is as predicted. \square

5.2 Elliptic Quartics

In this section we will compute walls in the case of *elliptic quartic curves* C in \mathbb{P}^3 , i.e. complete intersections of two quadrics inside \mathbb{P}^3 . The content of this section is based on [GHS16].

5.2.1 Tilt Stability

We will compute all the walls in tilt stability for $\beta < 0$ for the class $\text{ch}(\mathcal{I}_C)$. There is a locally free resolution $0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}(-2)^{\oplus 2} \rightarrow \mathcal{I}_C \rightarrow 0$. This leads to

$$\text{ch}^\beta(\mathcal{I}_C) = \left(1, -\beta, \frac{\beta^2}{2} - 4, -\frac{\beta^3}{6} + 4\beta + 8 \right).$$

Theorem 5.2.1. *There are three walls for $M_{\alpha,\beta}^{\text{tilt}}(1, 0, -4, 8)$ for $\alpha > 0$ and $\beta < 0$. The following table lists pairs of tilt semistable objects whose extensions completely describe all strictly semistable objects at each of the corresponding walls. Let L be a line in \mathbb{P}^3 , V a plane in \mathbb{P}^3 , $Z \subset \mathbb{P}^3$ a length two zero dimensional subscheme, $Z' \subset V$ a length two zero dimensional subscheme and $P \in \mathbb{P}^3$, $Q \in V$ be points.*

$\alpha^2 + (\beta + 3)^2 = 1$	$\mathcal{O}(-2)^{\oplus 2}, \mathcal{O}(-4)[1]$
$\alpha^2 + \left(\beta + \frac{7}{2}\right)^2 = \frac{17}{4}$	$\mathcal{I}_L(-1), \mathcal{O}_V(-3)$
$\alpha^2 + \left(\beta + \frac{9}{2}\right)^2 = \left(\frac{7}{2}\right)^2$	$\mathcal{I}_Z(-1), \mathcal{O}_V(-4)$
	$\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)$
	$\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)$

Table 5.2: Objects defining walls for elliptic quartics

The hyperbola $\nu_{\alpha,\beta}(1, 0, -4) = 0$ is given by the equation

$$\beta^2 - \alpha^2 = 8.$$

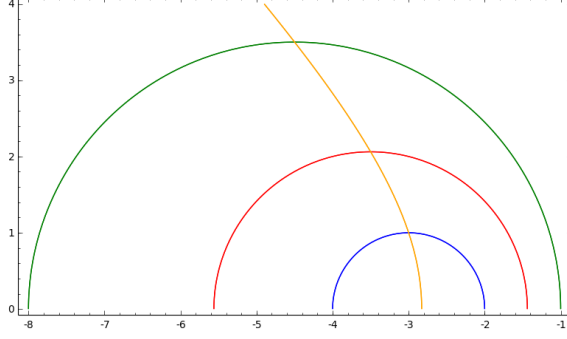


Figure 5.3: Walls in tilt stability for elliptic quartics

The first wall was already determined in much more generality in Theorem 4.3.1. Note that elliptic quartics are covered by this result with $n = -4$, $m = -2$, $i = 2$ and $j = 1$. In order to prove Theorem 5.2.1 we need to put numerical restrictions on potentially destabilizing objects. The next lemma determines the Chern characters of possibly destabilizing objects for $\beta = -2$.

Lemma 5.2.2. *If an exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ in $\text{Coh}^{-2}(\mathbb{P}^3)$ defines a wall for $\beta = -2$ with $\text{ch}_{\leq 2}^{-2}(E) = (1, 0, -4)$ then*

$$\text{ch}_{\leq 2}^{-2}(F), \text{ch}_{\leq 2}^{-2}(G) \in \left\{ \left(1, 1, -\frac{1}{2} \right), \left(0, 1, -\frac{3}{2} \right), \left(1, 1, \frac{1}{2} \right), \left(0, 1, -\frac{5}{2} \right) \right\}.$$

Proof. The argument is numerical and completely independent of exchanging quotient and subobject. The four possible Chern characters group into two cases, that add up to $\text{ch}_{\leq 2}^{-2}(E) = (1, 2, -2)$.

Let $\text{ch}_{\leq 2}^{-2}(F) = (r, c, d)$. By definition of $\text{Coh}^{-2}(\mathbb{P}^3)$, we have $0 \leq c \leq 2$. If $c = 0$, then $\nu_{\alpha, -2}(F) = \infty$ and this is in fact no wall for any $\alpha > 0$. If $c = 2$, then the same argument for the quotient G shows there is no wall. Therefore, $c = 1$ must hold. We

can compute

$$\nu_{\alpha,-2}(E) = -1 - \frac{\alpha^2}{4}, \quad \nu_{\alpha,-2}(F) = d - \frac{r\alpha^2}{2}.$$

The wall is defined by $\nu_{\alpha,-2}(E) = \nu_{\alpha,-2}(F)$. This leads to

$$\alpha^2 = \frac{4d+4}{2r-1} > 0. \tag{5.2}$$

The next step is to rule out the cases $r \geq 2$ and $r \leq -1$. If $r \geq 2$, then $\text{rk}(G) \leq -1$. By exchanging the roles of F and G in the following argument, it is enough to deal with the situation $r \leq -1$. In that case we use (5.2) and the Bogomolov Gieseker inequality to get the contradiction $2rd \leq 1$, $d < -1$ and $r \leq -1$.

Therefore, we know $r = 0$ or $r = 1$. By again interchanging the roles of F and G if necessary we only have to handle the case $r = 1$. Equation (5.2) implies $d > -1$. By Lemma 5.0.1 we get $d - 1/2 \in \mathbb{Z}_{\leq 0}$. Therefore, we are left with the cases in the claim. \square

Proof of Theorem 5.2.1. Since we are only dealing with $\beta < 0$ the structure theorem for walls in tilt stability implies that all walls intersect the left branch of the hyperbola. By Theorem 4.3.1 the smallest wall is already determined. This semicircle intersects the β -axis at $\beta = -4$ and $\beta = -2$. Therefore, all other walls intersecting this branch of the hyperbola also have to intersect the ray $\beta = -2$. By Lemma 5.2.2 there are at most two walls on this ray. They correspond to the two solution claimed to exist.

Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ define a wall in $\text{Coh}^{-2}(\mathbb{P}^3)$ with $\text{ch}(E) = (1, 0, -4, 8)$. One can compute $\text{ch}^{-2}(E) = (1, 2, -2, \frac{4}{3})$. A direct computation shows that the middle wall is given by $\text{ch}^{-2}(F) = (1, 1, -1/2, e)$ and $\text{ch}^{-2}(G) = (0, 1, -3/2, 4/3 - e)$. By Lemma 5.0.1 we get $F \cong \mathcal{I}_L(-1)$ where L is a line plus $1/6 - e$ (possibly embedded)

points in \mathbb{P}^3 . In particular, the inequality $e \leq 1/6$ holds. The same lemma also implies that $G \cong I_{Z/V}(-3)$ where Z is a dimension zero subscheme of length $e - 1/6$. Overall this shows $e = 1/6$. Therefore, L is just a line and $E \cong \mathcal{O}_V(-3)$.

The biggest wall is given by the twisted Chern characters $\text{ch}^{-2}(F) = (1, 1, 1/2, e)$ and $\text{ch}^{-2}(G) = (0, 1, -5/2, 4/3 - e)$. We use again Lemma 5.0.1 to get $F \cong \mathcal{I}_Z(-1)$ for a zero dimensional subscheme $Z \subset \mathbb{P}^3$ of length $1/6 - e$. Therefore, we have $e - 1/6 \in \mathbb{Z}_{\geq 0}$. The lemma also shows $G \cong I_{Z/V}(-4)$ where Z is a dimension zero subscheme of length $e + 11/6$. Overall, we get $e \in \{-11/6, -5/6, 1/6\}$. That corresponds exactly to the three cases in the Theorem. \square

5.2.2 Some Equations

Let $H_1 \subset \text{Hilb}(\mathbb{P}^3)$ be the closure of the locus of elliptic quartic curves. By $H_2 \subset \text{Hilb}(\mathbb{P}^3)$ we denote the closure of the locus of plane quartics curves plus two disjoint points. Notice that all curves in H_1 and H_2 have the same Hilbert polynomial and therefore lie on the same connected component of the Hilbert scheme.

Proposition 5.2.3. *I_C be the ideal of a one dimensional subscheme $C \subset \mathbb{P}^3$ that fits into an exact sequence of the form $0 \rightarrow \mathcal{I}_Z(-1) \rightarrow I_C \rightarrow \mathcal{O}_V(-4) \rightarrow 0$ where V is a plane in \mathbb{P}^3 and $Z \subset V$ is a zero dimensional subscheme of length two.*

(i) *The ideal I_C is projectively equivalent to one of the ideals*

$$(x^2, xy, xzw, f_4(x, y, z, w)),$$

$$(x^2, xy, xz^2, f_4(x, y, z, w)),$$

where $f_4 \in (x, y, zw)$, respectively $f_4 \in (x, y, z^2)$ is of degree 4.

(ii) If I_C is lying in a closed orbit of the Hilbert scheme under the action of $\mathrm{PGL}(4)$, then I_C is projectively equivalent to

$$(x^2, xy, xz^2, y^4).$$

Proof. Up to the action of $\mathrm{PGL}(4)$ we can assume that $I_Z = (x, y, zw)$ or $I_Z = (x, y, z^2)$ and $I_V = (x)$. The exact sequence $0 \rightarrow \mathcal{I}_Z(-1) \rightarrow I_C \rightarrow \mathcal{O}_V(-4) \rightarrow 0$ implies $(x^2, xy, xzw) \subset I_C$ or $(x^2, xy, xz^2) \subset I_C$. Since the quotient is $\mathcal{O}_V(-4)$, there has to be another degree 4 generator $f_4(x, y, z, w)$ with $xf_4(x, y, z, w) \in (x^2, xy, xzw)$ respectively $xf_4(x, y, z, w) \in (x^2, xy, xz^2)$. That proves (i).

Let I_C be an ideal lying in a closed orbit of the Hilbert scheme. By (i) we can assume $I_C = (x^2, xy, xzw, f_4(x, y, z, w))$ for a degree 4 polynomial $f_4 \in (x, y, zw)$ or $I_C = (x^2, xy, xz^2, f_4(x, y, z, w))$ for $f_4 \in (x, y, z^2)$. We can take the limit $t \rightarrow 0$ for the action of the element $g_t \in \mathrm{PGL}(4)$ that fixes x, y, z and maps $w \mapsto (1-t)z+tw$. Since the orbit is closed we can assume that $I_C = (x^2, xy, xz^2, f_4(y, z))$ where $f_4 \in \mathbb{C}[y, z]$. Pick $\lambda \in \mathbb{C}$ such that $f(\lambda, 1) \neq 0$. We analyze the action of $g_t \in \mathrm{PGL}(4)$ that fixes x, w , maps $y \mapsto \lambda y$ and maps $z \mapsto (1-t)y + tz$. We get

$$g_t \cdot (x^2, xy, xz^2, f_4(y, z)) = (x^2, xy, xz^2, f_4(\lambda y, (1-t)y + tz)).$$

Since $f(\lambda, 1) \neq 0$, we have $f_4(\lambda y, y) \neq 0$ and we can finish the proof of (ii) by taking the limit $t \rightarrow 0$. □

Next we want to analyze the singularities of the point on the Hilbert scheme corresponding to (x^2, xy, xz^2, y^4) . We will use [GS] and the techniques developed in [PS85].

Proposition 5.2.4. *If $I_C = (x^2, xy, xz^2, y^4)$, then C lies on the intersection of two components of $\text{Hilb}(\mathbb{P}^3)$ and is a smooth point on each of them. Moreover, the intersection is locally of dimension 15 and transversal.*

Proof. We use the Comparison Theorem [PS85, p. 764] which claims the Hilbert scheme $\text{Hilb}(\mathbb{P}^3)$ and the universal deformation space which parametrizes all homogeneous ideals with Hilbert function equal to that of I_C are isomorphic in an étale neighborhood of the point parametrizing C if

$$\left(\frac{\mathbb{C}[x, y, z, w]}{I_C} \right)_d \cong H^0(C, \mathcal{O}_C(d)) = 0$$

for all $d = \deg(f_i)$ where the f_i are generators of I_C . For our particular ideal this equality is true. The Comparison Theorem allows us to find local equations of the Hilbert scheme near C by using the same strategy as in the proof of [PS85, Lemma 6]. In fact, this procedure has been implemented in the Macaulay2 Package “VersalDeformations”. The routine `localHilbertScheme` generates an ideal in $\mathbb{C}[t_1, \dots, t_{24}]$ given by (see Appendix A)

$$(-t_5t_{24}, -t_6t_{24}, -t_7t_{24}, -t_8t_{24}, t_{15}t_{24}, t_{16}t_{24}, t_{17}t_{24} - 2t_{22}t_{24}, t_{18}t_{24} - 2t_{23}t_{24})$$

Then, locally at C , the Hilbert scheme is the transversal intersection of the hyperplane ($t_{24} = 0$) and a 16-dimensional linear subspace. □

It is not hard to see that the two components (x^2, xy, xz^2, y^4) is lying on are H_1 and H_2 by giving explicit degenerations. We skip this, since we will not need it and it will follow for free from the results in the next section.

5.2.3 Bridgeland Stability

The goal of this section is to translate the computations in tilt stability to actual wall crossings in Bridgeland stability. We will analyze the singular loci of moduli spaces on the path and use this to reprove the global description of the main component of the Hilbert scheme as in [VA92].

As an immediate consequence of Theorem 4.1.1 and Theorem 5.2.1 we obtain the following corollary. In this application of Theorem 4.1.1 all exact sequence giving walls in tilt stability left of the unique vertical wall are of the form in (iv). Therefore, we do not have more sequences giving walls in tilt stability than in Bridgeland stability to the left of the left branch of the hyperbola.

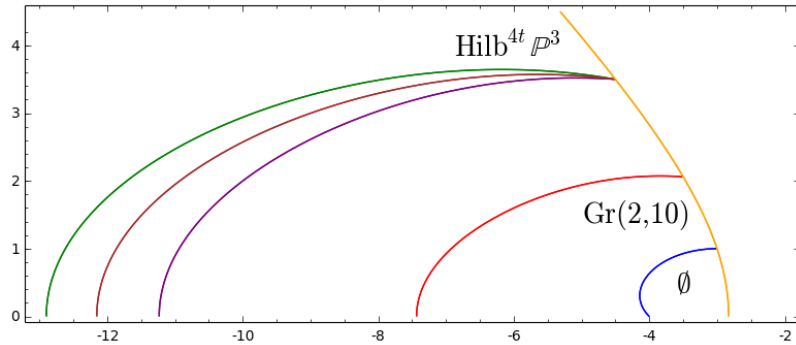


Figure 5.4: Walls in Bridgeland stability for elliptic quartics

Corollary 5.2.5. *There is a path $\gamma : [0, 1] \rightarrow \mathbb{R}_{>0} \times \mathbb{R} \subset \text{Stab}(\mathbb{P}^3)$ that crosses the following walls for $v = (1, 0, -4, 8)$ in the following order. The walls are defined by the two given objects having the same slope. Moreover, all strictly semistable objects at each of the walls are extensions of those two objects. Let L be a line in \mathbb{P}^3 , V a*

plane in \mathbb{P}^3 , $Z \subset \mathbb{P}^3$ a length two zero dimensional subscheme, $Z' \subset V$ a length two zero dimensional subscheme and $P \in \mathbb{P}^3$, $Q \in V$ be points.

(i) $\mathcal{O}(-2)^{\oplus 2}, \mathcal{O}(-4)[1]$

(ii) $\mathcal{I}_L(-1), \mathcal{O}_V(-3)$

(iii) $\mathcal{I}_Z(-1), \mathcal{O}_V(-4)$

(iv) $\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)$

(v) $\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)$

We denote the moduli space of Bridgeland stable objects with Chern character $(1, 0, -4, 8)$ in the chambers from inside the smallest wall to outside the largest wall by M_0, \dots, M_5 . The goal of this section is to give some description of these spaces. By Theorem 4.3.1 we have $M_0 = \emptyset$. After the largest wall we must have $M_5 = \text{Hilb}^{4t}(\mathbb{P}^3)$.

Proposition 5.2.6. *The first moduli space M_1 is the Grassmannian $G(2, 10)$.*

Proof. All extensions in $\text{Ext}^1(\mathcal{O}(-4)[1], \mathcal{O}(-2)^{\oplus 2})$ are cokernels of morphisms of the form $\mathcal{O}(-4) \rightarrow \mathcal{O}(-2)^{\oplus 2}$. The stability condition ensures that the two quadrics defining this equation are not collinear. Therefore, these extensions parametrize pencils of quadrics and the moduli space has to be the Grassmannian $G(2, 10)$. \square

The tangent space of a moduli space of Bridgeland stable objects at any stable complex E is given by $\text{Ext}^1(E, E)$. Obtaining these groups requires a substantial amount of diagram chasing and computations. In order to minimize the distress on the reader and the author, we will prove the following lemma with heavy usage of [GS].

Lemma 5.2.7. *Let notation be as in Theorem 5.2.5. The equalities*

$$\begin{aligned}
& \text{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3)) = \mathbb{C}, \quad \text{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1)) = \mathbb{C}^9, \\
& \text{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4, \quad \text{Ext}^1(\mathcal{O}_V(-3), \mathcal{O}_V(-3)) = \mathbb{C}^3, \\
& \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_V(-4)) = \begin{cases} \mathbb{C} & , Z \subset V \\ 0 & , \text{otherwise} \end{cases}, \quad \text{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_Z(-1)) = \mathbb{C}^{15}, \\
& \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_Z(-1)) = \mathbb{C}^6, \quad \text{Ext}^1(\mathcal{O}_V(-4), \mathcal{O}_V(-4)) = \mathbb{C}^3, \\
& \text{Ext}^1(\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)) = \begin{cases} \mathbb{C}^3 & , P = Q \\ \mathbb{C} & , P \neq Q \end{cases}, \\
& \text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1)) = \begin{cases} \mathbb{C}^{17} & , P = Q \\ \mathbb{C}^{15} & , P \neq Q \end{cases}, \\
& \text{Ext}^1(\mathcal{I}_P(-1), \mathcal{I}_P(-1)) = \mathbb{C}^3, \quad \text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_{Q/V}(-4)) = \mathbb{C}^5, \\
& \text{Ext}^1(\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)) = \mathbb{C}^2, \quad \text{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{O}(-1)) = \mathbb{C}^{15}, \\
& \text{Ext}^1(\mathcal{O}(-1), \mathcal{O}(-1)) = 0, \quad \text{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{I}_{Z'/V}(-4)) = \mathbb{C}^7
\end{aligned}$$

hold. If $Z \subset V$ is a double point supported at P , then

$$\begin{aligned}
& \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^3, \quad \text{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^2, \\
& \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_P(-1)) = \mathbb{C}^3, \quad \text{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_P(-1)) = \mathbb{C}^{15}.
\end{aligned}$$

Proof. Up to the action of $PGL(4)$ there are two orbits of pairs of a line and a plane (L, V) . Either we have $L \subset V$ or not. By choosing representatives defined over \mathbb{Q} , we can use [GS] to compute $\text{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3)) = \mathbb{C}$, $\text{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1)) = \mathbb{C}^9$, $\text{Ext}^1(\mathcal{O}_V(-3), \mathcal{O}_V(-3)) = \mathbb{C}^3$ and $\text{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4$. All other equalities follow in the same way. The Macaulay2 code can be found in Appendix A. \square

Since the dimension of tangent spaces is bounded from below by the dimension of the space, the following lemma can sometimes simplify the computations.

Lemma 5.2.8. *Let $0 \rightarrow F^n \rightarrow E \rightarrow G^m \rightarrow 0$ be an exact sequence at a wall in Bridgeland stability where F and G are distinct stable objects of the same Bridgeland slope and E is semistable to one side of the wall. Then the inequality*

$$\mathrm{ext}^1(E, E) \leq n^2 \mathrm{ext}^1(F, F) + m^2 \mathrm{ext}^1(G, G) + nm \mathrm{ext}^1(F, G) + nm \mathrm{ext}^1(G, F) - n^2$$

holds.

Proof. Stability to one side of the wall implies $\mathrm{Hom}(E, F) = 0$. Since F is stable, we also know $\mathrm{Hom}(F, F) = \mathbb{C}$. By the long exact sequence coming from applying $\mathrm{Hom}(\cdot, F)$ to the exact sequence for E , we get $\mathrm{ext}^1(E, F) \leq m \mathrm{ext}^1(G, F) + n \mathrm{ext}^1(F, F) - n$. Moreover, we can use $\mathrm{Hom}(\cdot, G)$ to get $\mathrm{ext}^1(E, G) \leq m \mathrm{ext}^1(G, G) + n \mathrm{ext}^1(F, G)$. These two inequalities together with applying $\mathrm{Hom}(E, \cdot)$ lead to the claim. \square

We also have to handle the issue of potentially new components after crossing a wall. The following result will solve this issue in some cases.

Lemma 5.2.9. *Let M and N be two moduli spaces of Bridgeland semistable objects separated by a single wall. Assume that $A \subset M$ and $B \subset N$ are the loci destabilized at the wall. If A intersects an irreducible component H of M non trivially and H is not contained in A , then B must intersect the closure of $H \setminus A$ inside N .*

Proof. This follows from the fact that moduli spaces of Bridgeland semistable objects are universally closed. If B would not intersect the closure of $H \setminus A$ inside N , then this would correspond to a component in N that is not universally closed. \square

In order to identify the global structure of some of the moduli spaces as blow ups we need the following classical result by Moishezon.

Theorem 5.2.10 ([Moi67]). *Any birational morphism $f : X \rightarrow Y$ between smooth proper algebraic spaces of finite type over \mathbb{C} such that the contracted locus E is irreducible and the image $f(E)$ is smooth is the blow up of Y in $f(E)$.*

Proposition 5.2.11. *The second moduli space M_2 is the blow up of $G(2, 10)$ in the smooth locus $G(2, 4) \times (\mathbb{P}^3)^\vee$ parametrizing pairs $(\mathcal{I}_L(-1), \mathcal{O}_V(-3))$.*

Proof. We know that M_1 is smooth. The wall separating M_1 and M_2 has strictly semistable objects given by extensions between $\mathcal{I}_L(-1)$ and $\mathcal{O}_V(-3)$. By Lemma 5.2.7 we have the equalities $\text{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3)) = \mathbb{C}$, $\text{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1)) = \mathbb{C}^9$, $\text{Ext}^1(\mathcal{O}_V(-3), \mathcal{O}_V(-3)) = \mathbb{C}^3$ and $\text{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4$.

This means the locus of extensions in $\text{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3))$ is isomorphic to $G(2, 4) \times (\mathbb{P}^3)^\vee$, i.e. is smooth and irreducible. By Lemma 5.2.8 any extension E in $\text{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1))$ satisfies $\text{ext}^1(E, E) \leq 16$. Lemma 5.2.9 shows that M_2 has to be smooth and irreducible. The locus of extensions in $\text{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1))$ is irreducible of dimension 15, i.e. is a divisor in M_2 . An immediate application of Theorem 5.2.10 implies the fact that M_2 is the blow up of $G(2, 10)$ in the smooth locus $G(2, 4) \times (\mathbb{P}^3)^\vee$. \square

The next moduli space will acquire a second component. This makes the technicalities more complicated.

Proposition 5.2.12. *The third moduli space M_3 has two irreducible components M_3^1 and M_3^2 . The first component M_3^1 is the blow up of M_2 in the smooth incidence variety parametrizing length two subschemes in a plane in \mathbb{P}^3 . The second component M_3^2 is a \mathbb{P}^{14} -bundle over $\text{Hilb}^2(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$ parametrizing pairs $(\mathcal{I}_Z(-1), \mathcal{O}_V(-4))$. The two components intersect transversally in the exceptional locus of the blow up.*

Proof. By Lemma 5.2.7 we have

$$\begin{aligned} \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_V(-4)) &= \begin{cases} \mathbb{C} & , Z \subset V \\ 0 & , \text{otherwise,} \end{cases} \\ \text{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_Z(-1)) &= \mathbb{C}^{15}. \end{aligned}$$

This means the locus destabilized in M_2 is of dimension 7 and the new locus appearing in M_3 is of dimension 23. Since M_2 is of dimension 16, the locus appearing in M_3 must be a new component M_3^2 . The closure of what is left of M_2 is denoted by M_3^1 . If M_3^2 is reduced, it is a \mathbb{P}^{14} -bundle over $\text{Hilb}^2(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$ parametrizing pairs $(\mathcal{I}_Z(-1), \mathcal{O}_V(-4))$. We will more strongly show that it is smooth.

Assume Z is not scheme theoretically contained in V . Then Lemma 5.2.8 implies that any non-trivial extension E in $\text{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_Z(-1))$ satisfies $\text{ext}^1(E, E) \leq 23$. Therefore, it is a smooth point and can in particular not lie on M_3^1 . Let E be an extensions of the form $0 \rightarrow \mathcal{I}_Z(-1) \rightarrow E \rightarrow \mathcal{O}_V(-4) \rightarrow 0$, where $Z \subset V$. Any point on the intersection must satisfy $\text{ext}^1(E, E) \geq 24$. Assume E is not an ideal sheaf. If E fits into an exact sequence $0 \rightarrow \mathcal{I}_{Z/V}(-4) \rightarrow E \rightarrow \mathcal{O}(-1) \rightarrow 0$ or $0 \rightarrow \mathcal{I}_{Q/V}(-4) \rightarrow E \rightarrow \mathcal{I}_P(-1) \rightarrow 0$ for $P \neq Q$, then a direct application of Lemma 5.2.8 to these sequences shows $\text{ext}^1(E, E) \leq 23$, a contradiction. Therefore, E must fit into an exact sequence $0 \rightarrow \mathcal{I}_{P/V}(-4) \rightarrow E \rightarrow \mathcal{I}_P(-1) \rightarrow 0$. Then we have the following commutative diagram with short exact rows and columns.

$$\begin{array}{ccccc} 0 & \hookrightarrow & \mathcal{I}_{P/V}(-4) & \twoheadrightarrow & \mathcal{I}_{P/V}(-4) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_Z(-1) & \hookrightarrow & E & \twoheadrightarrow & \mathcal{O}_V(-4) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_Z(-1) & \hookrightarrow & \mathcal{I}_P(-1) & \twoheadrightarrow & \mathcal{O}_P \end{array}$$

Therefore, Z has to be a double point supported at P . By Lemma 5.2.7 we have

$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^3, \quad \mathrm{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^2$$

$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_P(-1)) = \mathbb{C}^3, \quad \mathrm{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_P(-1)) = \mathbb{C}^{15}.$$

Next, we apply $\mathrm{Hom}(\cdot, \mathcal{I}_{P/V}(-4))$ to $0 \rightarrow \mathcal{I}_Z(-1) \rightarrow E \rightarrow \mathcal{O}_V(-4) \rightarrow 0$ to get $\mathrm{ext}^1(E, \mathcal{I}_{P/V}(-4)) \leq 5$. By applying $\mathrm{Hom}(\cdot, \mathcal{I}_P(-1))$ to the same sequence we get $\mathrm{ext}^1(E, \mathcal{I}_P(-1)) \leq 18$. Finally, we can use $\mathrm{Hom}(E, \cdot)$ on $0 \rightarrow \mathcal{I}_{P/V} \rightarrow E \rightarrow \mathcal{I}_Q(-1) \rightarrow 0$ to get $\mathrm{ext}^1(E, E) \leq 23$.

Therefore, the intersection of M_3^1 and M_3^2 parametrizes ideals fitting into an exact sequence $0 \rightarrow \mathcal{I}_Z(-1) \rightarrow I_C \rightarrow \mathcal{O}_V(-4) \rightarrow 0$, where $Z \subset V$. The intersection must have a closed orbit, but by Proposition 5.2.3, there is a unique closed orbit of this form. If the intersection were disconnected, it would have at least two closed orbits. If it is reducible, then the closed orbit must lie on the intersection of all irreducible components. By Proposition 5.2.4 the intersection at the closed orbit is transversal of dimension 15 and its points are smooth on both components. That would be impossible if the intersection is not irreducible at the closed orbit. The singular locus on either component is closed and must therefore contain a closed orbit. Thus, the whole intersection must consist of points that are smooth on each of the two components individually. The induced map $M_3^1 \rightarrow M_2$ contracts the intersection, which is an irreducible divisor, onto a locus isomorphic to the smooth incidence variety parametrizing length two subschemes in a plane in \mathbb{P}^3 . Theorem 5.2.10 implies the description of M_3^1 . \square

In order to reprove the description of the main component of the Hilbert scheme from [VA92], we have to make sure that none of the remaining walls modify the first component.

Proposition 5.2.13. *The fourth moduli space M_4 has two irreducible components M_4^1 and M_4^2 . The first component equal to M_3^1 . The second component is birational to M_3^2 .*

Proof. Lemma 5.2.7 says

$$\begin{aligned} \text{Ext}^1(\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)) &= \begin{cases} \mathbb{C}^3 & , P = Q \\ \mathbb{C} & , P \neq Q \end{cases} , \\ \text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1)) &= \begin{cases} \mathbb{C}^{17} & , P = Q \\ \mathbb{C}^{15} & , P \neq Q \end{cases} . \end{aligned}$$

Moreover, the moduli space of pairs $(\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4))$ is irreducible of dimension 8, while the sublocus where $P = Q$ is of dimension 5. Therefore, the closure of the locus of extensions in $\text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1))$ for $P \neq Q$ is irreducible of dimension 22. The locus of extensions in $\text{Ext}^1(\mathcal{I}_{P/V}(-4), \mathcal{I}_P(-1))$ for $P \in V$ is irreducible of dimension 21. Let M_4^1 be the closure of what is left from M_3^1 in M_4 and M_4^2 be the closure of what is left from M_3^2 .

If $P \neq Q$, then Lemma 5.2.8 implies smoothness. In particular, we can use Lemma 5.2.9 to show that all points in $\text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1))$ for $P \neq Q$ are in M_4^2 and no other component. Assume we have a general non trivial extension $0 \rightarrow \mathcal{I}_P(-1) \rightarrow E \rightarrow \mathcal{I}_{P/V}(-4) \rightarrow 0$. Then $E = I_C$ is an ideal sheaf of a plane quartic curve plus a double point in the plane. We can assume that the double point is not an embedded point due to the fact that E is general. Clearly, I_C is the flat limit of elements in $\text{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1))$ by choosing $P \notin V$ and regard the limit $P \rightarrow Q$. Therefore, E is a part of M_4^2 .

We showed $M_4 = M_4^1 \cup M_4^2$ and that M_4^2 is birational to M_3^2 . We are left to show $M_4^1 = M_4^2$. If not, there is an object E with a non trivial exact sequence $0 \rightarrow \mathcal{I}_P(-1) \rightarrow E \rightarrow \mathcal{I}_{P/V}(-4) \rightarrow 0$ in M_4^1 . By Lemma 5.2.9 this implies that there is also an object E' with non trivial exact sequence $0 \rightarrow \mathcal{I}_{P/V}(-4) \rightarrow E' \rightarrow \mathcal{I}_P(-1) \rightarrow 0$ lying on M_3^1 . But we already established that all those extensions are smooth points on M_3^2 in the previous proof. \square

We have mostly proven the following theorem.

Theorem 5.2.14. *The Hilbert scheme $\text{Hilb}^{4t}(\mathbb{P}^3)$ has two components H_1 and H_2 . The main component H_1 contains an open subset of elliptic quartic curves and is a smooth double blow up of the Grassmannian $G(2, 10)$. The second component is of dimension 23. Moreover, the two components intersect transversally in a locus of dimension 15.*

Proof. By Lemma 5.2.7 we have

$$\begin{aligned} \text{Ext}^1(\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)) &= \mathbb{C}^2, \\ \text{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{O}(-1)) &= \mathbb{C}^{15}, \\ \text{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{I}_{Z'/V}(-4)) &= \mathbb{C}^7. \end{aligned}$$

The moduli of objects $\mathcal{I}_{Z'/V}$ is irreducible of dimension 5. Lemma 5.2.8 implies that all strictly semistable objects at the largest wall except for the direct sum are smooth points on either M_4 or $M_5 = \text{Hilb}^{4t}(\mathbb{P}^3)$. Therefore, we can again use Lemma 5.2.9 to see that $\text{Hilb}^{4t}(\mathbb{P}^3)$ has exactly two components birational to M_4^1 and M_4^2 . Moreover, this argument shows that the ideals that destabilize at the largest wall cannot lie on the intersection of the two components and we have $M_5^1 = M_4^1$. \square

5.3 Computing Walls Algorithmically

The computational side in the previous examples is rather straightforward. In this section we discuss how this problem can be solved by computer calculations. This section is mostly based on [Sch15].

The proof of the following Lemma provides useful techniques for actually determining walls. As before X is a smooth projective threefold, H an ample polarization and for any $\alpha > 0$, $\beta \in \mathbb{R}$ we have a very weak stability condition $(\text{Coh}^\beta(X), Z_{\alpha,\beta}^{\text{tilt}})$.

Lemma 5.3.1. *Let $\beta \in \mathbb{Q}$ and v be the Chern character of some object of $D^b(X)$. Then there are only finitely many walls in tilt stability for this fixed β with respect to v .*

Proof. Any wall has to come from an exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ in $\text{Coh}^\beta(X)$. Let $H \cdot \text{ch}_{\leq 2}^\beta(E) = (R, C, D)$ and $H \cdot \text{ch}_{\leq 2}^\beta(F) = (r, c, d)$. Notice that due to the fact that $\beta \in \mathbb{Q}$ the possible values of r , c and d are discrete in \mathbb{R} . Therefore, it will be enough to bound those values to get finiteness.

By the definition of $\text{Coh}^\beta(X)$ one has $0 \leq c \leq C$. If $C = 0$, then $c = 0$ and we are dealing with the unique vertical wall. Therefore, we may assume $C \neq 0$. Let $\Delta := C^2 - 2RD$. The Bogomolov inequality together with Lemma 2.2.7 implies $0 \leq c^2 - 2rd \leq \Delta$. Therefore, we get

$$\frac{c^2}{2} \geq rd \geq \frac{c^2 - \Delta}{2}.$$

Since the possible values of r and d are discrete in \mathbb{R} , there are finitely many possible values unless $r = 0$ or $d = 0$. If $R \neq 0$ and $D \neq 0$, then using the same type of inequality for G instead of E will finish the proof.

Assume $R = r = 0$. Then the equality $\nu_{\alpha,\beta}(F) = \nu_{\alpha,\beta}(E)$ holds if and only if $Cd - Dc = 0$. In particular, it is independent of (α, β) . Therefore, the sequence does not define a wall.

Assume $D = d = 0$. Then the equality $\nu_{\alpha,\beta}(F) = \nu_{\alpha,\beta}(E)$ holds if and only if $Rc - Cr = 0$. Again this cannot define a wall. \square

Note that together with the structure theorem for walls in tilt stability this lemma implies that there is a biggest semicircle on each side of the vertical wall.

The proof of the Lemma tells us how to algorithmically solve the problem of determining all walls on a given vertical line. Assuming that β does not give the unique vertical wall, we have the following inequalities for any exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ defining a potential wall.

$$0 < H \cdot \text{ch}_1^\beta(F) < H \cdot \text{ch}_1^\beta(E),$$

$$0 < H \cdot \text{ch}_1^\beta(G) < H \cdot \text{ch}_1^\beta(E),$$

$$Q^{\text{tilt}}(F, F) \geq 0,$$

$$Q^{\text{tilt}}(G, G) \geq 0,$$

$$Q^{\text{tilt}}(F, G) \geq 0.$$

Moreover, we need $H \cdot \text{ch}(F)$ and $H \cdot \text{ch}(G)$ to be in the lattice spanned by Chern characters of objects in $D^b(X)$. Finally, the fact that the Chern classes of F and G are integers puts further restrictions on the possible values of the Chern characters. The code for a concrete implementation in [Dev15] can be found in Appendix B or on the authors website.

Appendix A: Macaulay2 Code

This is Macaulay2 code used in Proposition 5.2.4 and Lemma 5.2.7.

```

-----
-- Computation for Proposition 5.2.4 --
-----
needsPackage "VersalDeformations";
S = QQ[x,y,z,w];
F0 = matrix {{x^2,x*y,x*z^2,y^4}};
(F,R,G,C) = localHilbertScheme(F0,Verbose=>2);
T = ring first G;
sum G

-----
-- Computation for Lemma 5.2.7 --
-----
-- A = O.V(-3)
-- B = I.L(-1) L is not contained in V
-- C = I.L(-1) L is contained in V
X = Proj(QQ[x,y,z,w]);
A = (OO_X(0) / sheaf module ideal(x))*OO_X(-3);
B = (sheaf module ideal(y,z))*OO_X(-1);
C = (sheaf module ideal(x,y))*OO_X(-1);
Ext^1(B,A)
Ext^1(C,A)
Ext^1(A,B)
Ext^1(A,C)
Ext^1(A,A)
Ext^1(B,B)
Ext^1(C,C)

-----
-- A = O.V(-4)
-- B = I.Z(-1) Two separate point
--           outside V
-- C = I.Z(-1) Double point outside V
-- D = I.Z(-1) One point inside,
--           one point outside V
-- E = I.Z(-1) Two separate points
--           inside V
-- F = I.Z(-1) Double point scheme
--           theoretically in V
-- G = I.Z(-1) Double point set but
--           not scheme theoretically in V
X = Proj(QQ[x,y,z,w]);
A = (OO_X(0) /
  sheaf module ideal(x))*OO_X(-4);
B = (sheaf module ideal
  (y*(x-y),z,w))*OO_X(-1);
C = (sheaf module ideal(y^2,z,w))*OO_X(-1);
D = (sheaf module ideal(x*y,z,w))*OO_X(-1);
E = (sheaf module ideal(x,y*z,w))*OO_X(-1);
F = (sheaf module ideal(x,y,z^2))*OO_X(-1);
G = (sheaf module ideal(x^2,y,z))*OO_X(-1);
Ext^1(A,B)
Ext^1(A,C)
Ext^1(A,D)
Ext^1(A,E)
Ext^1(A,F)
Ext^1(A,G)
Ext^1(B,A)
Ext^1(C,A)
Ext^1(D,A)

-----
Ext^1(E,A)
Ext^1(F,A)
Ext^1(G,A)
Ext^1(A,A)
Ext^1(B,B)
Ext^1(C,C)

-----
-- A = I_{Q/V}(-4)
-- B = I.P(-1) P \notin V
-- C = I.P(-1) P \in V, P \neq Q
-- D = I.P(-1) P = Q
X = Proj(QQ[x,y,z,w]);
A = (sheaf module ideal(x,y,z)
  / sheaf module ideal(x))*OO_X(-4);
B = (sheaf module ideal(y,z,w))*OO_X(-1);
C = (sheaf module ideal(x,y,w))*OO_X(-1);
D = (sheaf module ideal(x,y,z))*OO_X(-1);
Ext^1(A,B)
Ext^1(A,C)
Ext^1(A,D)
Ext^1(B,A)
Ext^1(C,A)
Ext^1(D,A)
Ext^1(A,A)
Ext^1(B,B)
Ext^1(D,D)

-----
-- A = O(-1)
-- B = I_{Z^3/V}(-4) Two separate points
-- C = I_{Z^3/V}(-4) Double point
X = Proj(QQ[x,y,z,w]);
A = OO_X(-1);
B = (sheaf module ideal(x,y,z^2)
  / sheaf module ideal(x))*OO_X(-4);
C = (sheaf module ideal(x,y,w*z)
  / sheaf module ideal(x))*OO_X(-4);
Ext^1(B,A)
Ext^1(C,A)
Ext^1(A,B)
Ext^1(A,C)
Ext^1(B,B)
Ext^1(C,C)

-----
-- A = I.Z(-1), Z \subset V double point at P
-- B = O.V(-4)
-- C = I_{P/V}(-4)
-- D = I.P(-1)
X = Proj(QQ[x,y,z,w]);
A = (sheaf module ideal(x,y,z^2))*OO_X(-1);
B = (OO_X(0) / sheaf module ideal(x))*OO_X(-4);
C = (sheaf module ideal(x,y,z)
  / sheaf module ideal(x))*OO_X(-4);
D = (sheaf module ideal(x,y,z))*OO_X(-1);
Ext^1(A,C)
Ext^1(B,C)
Ext^1(A,D)
Ext^1(B,D)

```

Appendix B: Sage Code

The Sage [Dev15] code at the end of this appendix is a library for computations in tilt stability written by the author. A full documentation can be found on the authors website. The following piece of code can be used to verify and draw the walls in tilt stability for the case of twisted cubics.

```
var('a,b', domain = RR)
#The Chern character of an ideal sheaf of a twisted cubic in  $\mathbb{P}^3$ .
v = 3*O(-2) - 2*O(-3)
l = v.computeWalls(-2)
print l
J = (b, -6, -1)
K = (a, 0, 4)
p = implicit_plot(v.hyperbola(a, b), J, K, color = 'orange')
for w in l:
    p += implicit_plot(expand(v.wall(w, a, b, tilt = True)), J, K)
#All walls have to be outside this semicircle due to
#the generalized Bogomolov Gieseker inequality.
p += implicit_plot(v.Q(v, a, b, bridgeland = True), J, K, color = 'green')
show(p)
```

The next code can be used to verify and draw the walls in tilt stability for the case of elliptic quartics.

```
var('a,b', domain = RR)
#The Chern character of an ideal sheaf of a twisted cubic in  $\mathbb{P}^3$ .
v = 2*O(-2) - O(-4)
l = v.computeWalls(-2)
print l
J = (b, -8, -1)
K = (a, 0, 4)
p = implicit_plot(v.hyperbola(a, b), J, K, color = 'orange')
for w in l:
    p += implicit_plot(expand(v.wall(w, a, b, tilt = True)), J, K)
#All walls have to be outside this semicircle due to
#the generalized Bogomolov Gieseker inequality.
p += implicit_plot(v.Q(v, a, b, bridgeland = True), J, K, color = 'green')
show(p)
```

This is the library necessary to run the code above. It is strongly recommended that the reader who wants to run it downloads the latest version from the authors website.

```

"""
Created on 05 August 2015
@author: Benjamin Schmidt

Library for various computations on stability in  $P^3$ .

 $Z_{\{a, b, s, \text{bridgeland} = \text{True}\}} = -\text{ch}_3^b + (1/6 + s)*a^2*\text{ch}_1^b + I*(\text{ch}_2^b - a^2/2*\text{ch}_0^b)$ .

 $Z_{\{a, b, \text{tilt} = \text{True}\}} = -\text{ch}_2^b + a^2/2*\text{ch}_0^b + I*\text{ch}_1^b$ . slope = nu.

 $Z_{\{a, b\}} = -\text{ch}_1^b + I*\text{ch}_0^b$ . slope = mu."""

def isInt(n):
    """Returns True if n is an integer and False if n is not an integer."""
    if floor(n) - n == 0:
        return True
    return False

def inLattice(v, b, deg = 1):
    """Returns True if  $\text{ch}^{-b}(v) \in \mathbb{Z}^2 \oplus 1/2/d \mathbb{Z} \oplus \dots \oplus 1/i! \mathbb{Z}$ .
    False otherwise."""
    False otherwise."""
    w = [CohomologyClass(v).ch(i, -b) for i in range(len(v))]
    for i in range(len(w)):
        if i == 2:
            if isInt(w[i]*factorial(i)*deg) == False:
                return False
        else:
            if isInt(w[i]*factorial(i)) == False:
                return False
    return True

def mathRange(a, b, steps = 1, lopen = False, ropen = False):
    """Returns a list of numbers x between a and b such that  $x \in \text{steps}*\mathbb{Z}$ .
    If lopen = True, then it does not include a.
    If ropen = True, then it does not include b."""
    l = []
    if lopen == False:
        value = roundUp(a, steps)
    else:
        value = a + steps
    if ropen == True:
        while value < b:
            l.append(value)
            value += steps
    else:
        while value <= b:
            l.append(value)
            value += steps
    return l

def productRange(a, b, stepx = 1, stepy = 1, lopen = False, ropen = False):
    """Returns a list of pairs (x,y) such that  $x*y$  is in in between a and b,
    where  $x \neq 0 \neq y$ .
    Moreover,  $x \in \text{stepx}*\mathbb{Z}$  and  $y \in \text{stepy}*\mathbb{Z}$ .
    If lopen = True, then  $a < x*y$ . Otherwise,  $a \leq x*y$ .
    If ropen = True, then  $x*y < b$ . Otherwise,  $x*y \leq b$ ."""
    l = []
    m = max(abs(a), abs(b))/stepy
    for x in mathRange(-m, m, stepx, lopen, ropen):
        if x == 0:
            continue
        if x < 0:
            for y in mathRange(b/x, a/x, stepy, lopen, ropen):
                if y == 0:
                    continue
                l.append((x,y))
        if x > 0:
            for y in mathRange(a/x, b/x, stepy, lopen, ropen):
                if y == 0:
                    continue
                l.append((x,y))
    return l

def roundUp(a, steps = 1):
    """Returns  $\min\{x \in \text{steps}*\mathbb{Z} : a \leq x\}$ ."""
    #This is equivalent to  $\min\{x \in \mathbb{Z} : a/\text{steps} \leq x/\text{steps}\}$ 
    val = floor(a/steps)*steps
    if a <= val:
        return val
    return val + steps

def isList(l):
    """Returns True if l supports l[0]. Return False if not."""
    try:
        l[0]
        return True
    except:
        return False

```

```

def circleData(equation):
    """Takes the equation of a semicircle in tilt stability and puts it into some normal form.
    The expression needs to be of the form  $x*a^2 + x*b^2 + y*b + z$  with variables a and b."""
    eq = expand(equation)
    eq = expand(eq/eq.coefficient(b,2))
    A = eq.coefficient(b,1)
    B = eq.coefficient(b,0).coefficient(a,0)
    return a^2 + (b+(A/2).full_simplify())^2 + (B - A^2/4).full_simplify()

def xi(E, F):
    """Computes the Euler Characteristic xi(self, other)
    via the Hirzebruch–Riemann–Roch Theorem."""
    newE = CohomologyClass((E[0], -E[1], E[2], -E[3]))
    return (F*newE*CohomologyClass((1,2,11/6,1)))[3]

from operator import itemgetter

class CohomologyClass():
    """A cohomology class in  $H^*(P^3)$ . Elements can be accessed and changed as in a list.
    Operators: + - * / **"""
    def __init__(self, vec = (0,0,0,0)):
        self.bernCharacter = vector(vec)

    def Z(self, a=1, b=0, s=0, tilt = False, bridgeland = False):
        """Returns the central charge of the object. If tilt = True returns the tilt stability
        central charge. If bridgeland = True return the Bridgeland central charge. In all other
        cases returns the central charge for classical Mumford slope–stability."""
        if tilt == True:
            return -self.ch(2, b) + a^2/2*self.ch(0, b) + I*self.ch(1, b)
        if bridgeland == True:
            return (-self.ch(3, b) + (1/6+s)*a^2*self.ch(1, b) +
                I*(self.ch(2, b) - a^2/2*self.ch(0, b)))
        return -self.ch(1, b) + I*self.ch(0, b)

    def mu(self, a=1, b=0, s=0, tilt = False, bridgeland = False):
        """Returns the slope of the object. If tilt = True returns the nu–slope.
        If bridgeland = True returns the lambda–slope.
        In all other cases returns the classical Mumford slope."""
        centralCharge = self.Z(a, b, s, tilt, bridgeland)
        if centralCharge.imag() == 0:
            return oo
        return -centralCharge.real()/centralCharge.imag()

    def Q(self, v, a = 0, b = 0, K = 1, tilt = False, bridgeland = False):
        """Returns the value of the bilinear form  $Q_{\{a, b, K\}}(v, self)$  if bridgeland = True.
        If tilt = True, then it returns the mixed Bogomolov–Gieseker discriminant between v, w."""
        if tilt == True:
            return v[1]*self[1] - v[0]*self[2] - v[2]*self[0]
        if bridgeland == True:
            return (K*a^2*self.Q(v, tilt = True) + 4*v.ch(2, b)*self.ch(2, b)
                - 3*v.ch(1, b)*self.ch(3, b) - 3*v.ch(3, b)*self.ch(1, b))
        return 0

    def wall(self, v, a = 0, b = 0, s = 0, tilt = False, bridgeland = False):
        """Returns an expression whose zero set is the numerical wall between
        two objects with Chern character v and self. If tilt = True it is
        computed in tilt stability. If bridgeland = True
        it is computed in Bridgeland stability"""
        A = self.Z(a, b, s, tilt, bridgeland)
        B = v.Z(a, b, s, tilt, bridgeland)
        return A.real()*B.imag() - A.imag()*B.real()

    def hyperbola(self, a = 0, b = 0):
        """Returns an expression whose zero set the hyperbola  $\nu_{\{a, b\}}(self) = 0$ ."""
        return self.Z(a, b, tilt = True).real()

    def ch(self, l, b=0):
        """Returns the l–th Chern character twisted by b."""
        value = 0
        for j in range(l+1):
            value += (-b)^(l-j)/factorial(l-j)*self[j]
        return value

    def __getitem__(self, i):
        """Returns the i–th Chern character."""
        return self.bernCharacter[i]

    def __setitem__(self, i, c):
        """Change the i–th Chern character."""
        self.bernCharacter[i] = c

    def __len__(self):
        """Returns the length of self.bernCharacter. Usually 3."""
        return len(self.bernCharacter)

    def __str__(self):
        """Returns the class as a string."""

```

```

        return self.chernCharacter._str_()

def __repr__(self):
    """Returns the class as a string."""
    return self.chernCharacter._repr_()

def __add__(self, other):
    """Returns self + other."""
    return CohomologyClass(self.chernCharacter + other.chernCharacter)

def __sub__(self, other):
    """Returns self - other."""
    return CohomologyClass(self.chernCharacter - other.chernCharacter)

def __mul__(self, other):
    """Returns self*other. Multiplication is the cup product."""
    if isinstance(other, list):
        v = []
        for i in range(len(self)):
            value = 0
            for j in range(i+1):
                value += self[j]*other[i-j]
            v.append(value)
        return CohomologyClass(v)
    return CohomologyClass(self.chernCharacter*other)

def __rmul__(self, other):
    """Returns other*self."""
    return self*other

def __pow__(self, n):
    """Returns self^n for any integer n.
    BUG: Do not use variables with name xx1, xx2, xx3 or xx4 with this."""
    var('xx1,xx2,xx3,xx4')
    if n > 0:
        result = CohomologyClass((self[0], self[1], self[2], self[3]))
        for i in range(n - 1):
            result *= self
        return result
    elif n == 0:
        return CohomologyClass((1,0,0,0))
    elif n == -1:
        if self[0] == 0:
            raise ZeroDivisionError
        inverse = CohomologyClass((xx1,xx2,xx3,xx4))
        identity = self*inverse
        eqs = [identity.chernCharacter[0] == 1,
              identity.chernCharacter[1] == 0,
              identity.chernCharacter[2] == 0,
              identity.chernCharacter[3] == 0]
        solution = solve(eqs, [xx1,xx2,xx3,xx4])
        return CohomologyClass((solution[0][0].rhs(),
                                solution[0][1].rhs(),
                                solution[0][2].rhs(),
                                solution[0][3].rhs()))
    elif n < -1:
        return (self^(-1))^(-n)
    raise TypeError

def __neg__(self):
    """Returns -self."""
    return CohomologyClass(-self.chernCharacter)

def __pos__(self):
    """Returns +self."""
    return self

def __div__(self, other):
    """Returns self/other."""
    return self*other^(-1)

def computeDimVector(self, collection):
    """Returns the dimension vector that any object with this Chern
    character must have with respect to the exception collection
    if the object is in the quiver category."""
    var('x,y,v,w')
    eqs = []
    eqs.append(-x*collection[0][0] + y*collection[1][0]
               - v*collection[2][0] + w*collection[3][0] == self[0])
    eqs.append(-x*collection[0][1] + y*collection[1][1]
               - v*collection[2][1] + w*collection[3][1] == self[1])
    eqs.append(-x*collection[0][2] + y*collection[1][2]
               - v*collection[2][2] + w*collection[3][2] == self[2])
    eqs.append(-x*collection[0][3] + y*collection[1][3]
               - v*collection[2][3] + w*collection[3][3] == self[3])
    solution = solve(eqs, [x,y,v,w])
    return vector((solution[0][0].rhs(),
                  solution[0][1].rhs(),

```

```

        solution[0][2].rhs(),
        solution[0][3].rhs())

def computeWalls(self, b, deg = 1):
    """Returns a list of dimension vectors for all potential walls
    in tilt-stability for fixed b. deg = min{HK > 0: K any divisor}
    in case we use this on a different space than projective space.
    TREAT DEG != 1 WITH CAUTION!"""
    t1 = 1/b.denominator() # Generator of Z[b]
    t2 = t1^2/2/deg # Generator of Z[1/2, b^2/2]

    # Subobject will have Chern characters (r,c,d)
    R = self.ch(0, b)
    C = self.ch(1, b)
    D = self.ch(2, b)
    l = [] #Output list

    Delta = C^2 - 2*R*D
    # By Bogomolov Delta >= 0.
    if Delta < 0:
        return l

    if C < 0:
        R, C, D = -R, -C, -D

    if C == 0:
        c = 0
        # We need to have D >= 0 and R <= 0,
        # since tilt-stability is a very weak stability condition.
        if D < 0:
            R, C, D = -R, -C, -D
        if R > 0:
            raise Exception
        l = [(r,c,d) for r in mathRange(R, 0) for d in mathRange(0, D, steps = t2)]

    if C > 0:
        # c \in (0, C) by definition of \Coh^b
        for c in mathRange(0, C, steps = t1, lopen = True, ropen = True):
            # First handle the case d = 0. If D = 0, there is no wall for d = 0.
            # We use Bogomolov for the quotient to rule out walls.
            d = 0
            if D > 0:
                for r in mathRange(R - (C-c)^2/(2*D), R - ((C-c)^2-Delta)/(2*D)):
                    l.append((r,c,d))
            elif D < 0:
                for r in mathRange(R - ((C-c)^2-Delta)/(2*D), R - (C-c)^2/(2*D)):
                    l.append((r,c,d))

            # Next we handle the case r = 0. If R = 0, there is no wall for r = 0.
            # We use Bogomolov for the quotient to rule out walls.
            r = 0
            if R > 0:
                for d in mathRange(D - (C-c)^2/(2*R), D - ((C-c)^2-Delta)/(2*R), steps = t2):
                    l.append((r,c,d))
            elif R < 0:
                for d in mathRange(D - ((C-c)^2-Delta)/(2*R), D - (C-c)^2/(2*R), steps = t2):
                    l.append((r,c,d))

            # Now we make a finite list for r \neq 0, d \neq 0 with
            # Bogomolov for the subobject.
            for (r,d) in productRange((c^2 - Delta)/2, c^2/2, stepy = t2):
                l.append((r,c,d))

    # Check all classes are in the correct lattice
    l2 = []
    for (r,c,d) in l:
        if inLattice((r,c,d), b, deg):
            l2.append((r,c,d))

    # Check Bogomolov on the quotient for the list
    l3 = []
    for (r,c,d) in l2:
        if (C-c)^2 - 2*(R-r)*(D-d) >= 0:
            l3.append((r,c,d))

    # Check that there is there is a solution to the semicircle
    # for this fixed b.
    l4 = []
    for (r,c,d) in l3:
        if (R*c - C*r) != 0:
            if 2*(D*c - C*d)/(R*c - C*r) > 0:
                l4.append((r,c,d))

    # Create CohomologyClasses out of this
    l5 = []
    for element in l4:
        v = CohomologyClass(element)
        r = v.ch(0, -b)

```

```

        c = v.ch(1, -b)
        d = v.ch(2, -b)
        l5.append(CohomologyClass((r, c, d)))

# Check whether the second Chern class is an integer
l6 = []
for v in l5:
    if isInt(v[2] - v[1]^2/2):
        l6.append(v)

return l6

class O(CohomologyClass):
    """A line bundle on P^3."""
    def __init__(self, n):
        """Create O(n)"""
        self.chernCharacter = vector([1, n, n^2/2, n^3/6])

class LeftMutation(CohomologyClass):
    """A left mutation of two exceptional vector bundles."""
    def __init__(self, E, F):
        """Create the left mutation between E and F.
        It is defined by L_E(F) -> Hom(E, F) otimes E -> F."""
        self.E = E
        self.F = F
        self.homs = hom(E, F)
        self.chernCharacter = self.homs*E.chernCharacter - F.chernCharacter

class RightMutation(CohomologyClass):
    """A right mutation of two exceptional vector bundles."""
    def __init__(self, E, F):
        """Create the right mutation between E and F.
        It is defined by E -> Hom(E, F)^{vee} otimes F -> R_F(E)."""
        self.E = E
        self.F = F
        self.homs = hom(E, F)
        self.chernCharacter = self.homs*F.chernCharacter - E.chernCharacter

def hom(A, B):
    """Return the dimension of Hom(A, B). Only works if A and B are in the
    same full strong exceptional collection or both are line bundles."""
    if isinstance(A, O):
        if isinstance(B, O):
            n = B[1] - A[1]
            #Hom(A, B) = H^0(O(n))
            if n < 0:
                return 0
            return binomial(n+3, n)
        elif isinstance(B, LeftMutation):
            #B = L_E(F)
            #0 -> Hom(A, L_E(F)) -> Hom(A, E^{hom(E, F)}) -> Hom(A, F)
            #-> Ext^1(A, L_E(F)) = 0
            #The Ext^1 is zero, since A and L_E(F) are part of the
            #same full strong exceptional collection.
            return B.homs*hom(A, B.E) - hom(A, B.F)
        elif isinstance(B, RightMutation):
            #B = R_F(E)
            #0 -> Hom(A, E) -> Hom(A, F^{hom(E, F)}) -> Hom(A, R_F(E))
            #-> Ext^1(A, E) = 0
            #The Ext^1 is zero, since A and E are part of the same
            #full strong exceptional collection.
            return B.homs*hom(A, B.F) - hom(A, B.E)
        else:
            raise TypeError()
    elif isinstance(A, LeftMutation):
        #A = L_E(F)
        #0 -> Hom(F, B) -> Hom(E^{hom(E, F)}, B) -> Hom(L_E(F), B) -> Ext^1(F, B) = 0
        #The Ext^1 is zero, since F and B are part of the same
        #full strong exceptional collection.
        return A.homs*hom(A.E, B) - hom(A.F, B)
    elif isinstance(A, RightMutation):
        #A = R_F(E)
        #0 -> Hom(R_F(E), B) -> Hom(F^{hom(E, F)}, B) -> Hom(E, B)
        #-> Ext^1(R_F(E), B) = 0
        #The Ext^1 is zero, since R_F(E) and B are part of the same
        #full strong exceptional collection.
        return A.homs*hom(A.F, B) - hom(A.E, B)
    else:
        raise TypeError()

class ExceptionalCollection():
    """A full strong exceptional collection of vector bundles on P^3.
    Elements can be accessed as in a list."""
    def __init__(self, n = 0):
        """Creates the exceptional collection O(n), O(n+1), O(n+2), O(n+3)
        on P^3"""
        self.E = [O(n), O(n+1), O(n+2), O(n+3)]

```

```

        self.__resetComputations()

def __resetComputations(self):
    self.arrows = None
    self.relations = None
    self.higherRelations = None
    self.dual = None

def __getitem__(self, b):
    return self.E[b]

def __setitem__(self, b, c):
    self.E[b] = c

def __len__(self):
    return len(self.E)

def copy(self):
    """Return a copy of the collection."""
    new = ExceptionalCollection()
    for i in range(len(self.E)):
        new[i] = self[i]
    return new

def leftMutate(self, i):
    """Mutates (E_i, E_{i+1}) to (L_{E_i}(E_{i+1}), E_i)."""
    self.E[i], self.E[i+1] = LeftMutation(self.E[i], self.E[i+1]), self.E[i]
    self.__resetComputations()

def rightMutate(self, i):
    """Mutates (E_i, E_{i+1}) to (E_{i+1}, R_{E_{i+1}}(E_i))."""
    self.E[i], self.E[i+1] = RightMutation(self.E[i], self.E[i+1]), self.E[i+1]
    self.__resetComputations()

def __dualCollection(self):
    """Compute the dual exceptional collection."""
    self.dual = self.copy()
    for i in range(1, len(self.dual)):
        for j in range(len(self.dual)-1, i-1, -1):
            self.dual[j-1], self.dual[j] = (self.dual[j],
                                           RightMutation(self.dual[j-1], self.dual[j]))

def dualCollection(self):
    """Returns the dual exceptional collection."""
    if self.dual == None:
        self.__dualCollection()
    return self.dual.copy()

def getArrows(self):
    """Returns a tuple with the numbers of arrows."""
    if self.arrows == None:
        self.arrows = []
        for i in range(0, len(self) - 1):
            self.arrows.append(hom(self[i], self[i+1]))
    return tuple(self.arrows)

def getRelations(self):
    """Returns a tuple with the numbers of relations assuming this is
    the exceptional collection determining simple representations."""
    if self.relations == None:
        self.relations = []
        if self.dual == None:
            self.__dualCollection()
        for i in range(len(self) - 3, -1, -1):
            self.relations.append(hom(self.dual[i], self.dual[i+1])
                                 *hom(self.dual[i+1], self.dual[i+2])
                                 - hom(self.dual[i], self.dual[i+2]))
    return tuple(self.relations)

def getHigherRelations(self):
    """Returns the number of relations between relations assuming
    this is the exceptional collection determining simple representations.
    THE MATH BEHIND THIS FUNCTION IS SHAKY."""
    if self.higherRelations == None:
        if self.dual == None:
            self.__dualCollection()
        self.higherRelations = hom(self.dual[0], self.dual[3])
    return self.higherRelations

class QuiverRep():
    """A representation of a quiver given by a full strong exceptional
    collection on P^3."""
    def __init__(self, collection, dim):
        """collection = a full strong exceptional collection on P^3
        dim = dimension vector"""
        if len(collection) != len(dim):
            raise TypeError()
        self.dim = vector(dim)

```

```

self.collection = collection

def expectedDimension(self):
    """ Computes the expected dimension of the moduli space of
    quiver representation with dimension vector self.dim.
    TREAT 4-TERM COMPLEXES WITH CAUTION"""
    arrows = self.collection.getArrows()
    relations = self.collection.getRelations()
    higherRelations = self.collection.getHigherRelations()
    expectedDim = 0
    expectedDim += (self.dim[0]*self.dim[1]*arrows[0] +
                    self.dim[1]*self.dim[2]*arrows[1] +
                    self.dim[2]*self.dim[3]*arrows[2]) #Dim of vector space
    expectedDim -= (self.dim[0]*self.dim[2]*relations[0] +
                    self.dim[1]*self.dim[3]*relations[1]) #Relations
    expectedDim += self.dim[0]*self.dim[3]*higherRelations #Relations between relations
    expectedDim -= (self.dim[0]**2 + self.dim[1]**2 + self.dim[2]**2 +
                    self.dim[3]**2 - 1) #Dimension of the Group
    return expectedDim

def leftMutate(self, i):
    """ Mutates (E_i, E_{i+1}) to (L_{E_i}(E_{i+1}), E_i).
    Computes the new dimension vector assuming the object stays in
    the category.
    IT IS NOT CHECKED WHETHER THE OBJECT STAYS IN THE CATEGORY."""
    self.collection.leftMutate(i)
    #In K-Theory we have
    #[E] = -dim[0]*[E_0] + dim[1]*[E_1] - dim[2]*[E_2] + dim[3]*[E_3].
    #[E_{i+1}] = hom(E_i, E_{i+1}) [E_i] - [L_{E_i}(E_{i+1})]
    (self.dim[i], self.dim[i+1]) = (self.dim[i+1],
                                     -self.dim[i] + self.collection[i].homs * self.dim[i+1])

def rightMutate(self, i):
    """ Mutates (E_i, E_{i+1}) to (E_{i+1}, R_{E_{i+1}}(E_i)).
    Computes the new dimension vector assuming the object stays
    in the category.
    IT IS NOT CHECKED WHETHER THE OBJECT STAYS IN THE CATEGORY."""
    self.collection.rightMutate(i)
    #In K-Theory we have
    #[E] = -dim[0]*[E_0] + dim[1]*[E_1] - dim[2]*[E_2] + dim[3]*[E_3].
    #[E_i] = hom(E_i, E_{i+1}) [E_{i+1}] - [R_{E_{i+1}}(E_i)]
    (self.dim[i], self.dim[i+1]) = (-self.dim[i+1] + self.collection[i+1].homs * self.dim[i],
                                     self.dim[i])

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