University of Bonn

Mathematical Institute

Master's Thesis

Resolutions of some Schubert Varieties

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Hereby I declare that I have written this thesis by my own. Furthermore, I confirm that no other sources have been used than those specified in the thesis itself.
This thesis, in same or similar form, has not been available to any audit authority yet.

Contents

1 Introduction

A fundamental goal of the theory of Schubert varieties is a thorough understanding of the singularities. A first step is the description of the singular locus. A follow up is the construction of resolutions. A well known class of desingularizations of Schubert varieties, first introduced in [BS58], are the Bott-Samelson resolutions. They were extensively studied by Demazure in [Dem74] which included a description of their Chow ring and their canonical divisor. Zelivinsky constructed small resolutions in the case of Schubert varieties in Grassmannians and used them to compute Kazdhan-Lusztig polynomials (see [Zel83]).

Definition 1.1. A projective birational morphism of varieties $\pi : Y \to X$ is called IH-small (resp. IH-semismall) if for all $k > 0$ the following inequality holds.

$$\operatorname{codim}_X\{x \in X \mid \dim \pi^{-1}(x) = k\} > 2k \text{ (resp. } \geq 2k)$$

In addition, if Y is smooth, then π is called an IH-small (resp. IH-semismall) resolution.

Sankaran and Vanchinathan obtained similar results in Lagrangian and maximal isotropic Grassmannians in [SV94] and [SV95]. As it turns out, Bott-Samelson resolutions are generally not small.

Many classical results on Schubert varieties in Grassmannians can be generalized onto minuscule or cominuscule Schubert varieties (see Definition 5.1). Brion and Polo describe the singular locus of any Schubert variety in the (co)minuscule case in [BP99]. In [Per07] Perrin gives a complete classification of all small resolutions of minuscule Schubert varieties over $\mathbb{C}$. We use his techniques to classify all small resolutions of cominuscule Schubert varieties which are not minuscule over $\mathbb{C}$. This only includes semisimple groups with root system B_n or C_n. The construction itself will work over any algebraically closed field. Therefore, the result includes the small resolutions in maximal isotropic Grassmannians of [SV94].

In order to handle the occurring combinatorics in the Weyl group we introduce a quiver for each reduced expression (see Section 3.3). Due to a result in [Ste96], every reduced expression of a (co)minuscule element is unique up to commuting relations. That will imply that there is a unique quiver associated to each cominuscule Schubert variety. Moreover, there is an explicit combinatorial description of the quivers of (co)minuscule elements (see Lemma 5.5). That gives us a very concrete object to work with. We define a partial ordering on the vertices of the quiver, call the maximal elements peaks and assign each vertex a value we call the height.

Constructing small resolutions for cominuscule Schubert varieties $X_P(w)$ will be done by defining intermediate resolutions through which the Bott-Samelson resolution factors. An ordering of the peaks, or in some cases of the peaks and one additional vertex, yields a birational projective map $\widehat{\pi} : \widehat{X}(w) \to X_P(w)$. The variety $\widehat{X}(w)$ is generally not smooth, but always locally $\mathbb{Q}$-factorial with only terminal singularities. We introduce specific orderings that are called neat and kind of neat. The following theorem holds over any algebraically closed.

Theorem 1.2. *Let $X_P(w)$ be a cominuscule Schubert variety. Then $\widehat{\pi} : \widehat{X}(w) \to X_P(w)$ is a small resolution if $\widehat{X}(w)$ is smooth and obtained from a neat ordering.*

This is proven by a direct computation. Moreover, we show that this already describes all possible small resolution over $\mathbb{C}$. This can be done using a connection to the relative minimal model program. We use a result from Totaro in [Tot00]. Using a key result of [Wis91] he proves the following proposition.

Proposition 1.3. *Let $\pi : Y \to X$ be an IH-small resolution. Then Y is a relative minimal model of X.*

A relative minimal model of a variety X is a normal variety $\widehat{X}$ with a proper morphism $\widehat{X} \to X$ which is locally $\mathbb{Q}$-factorial with only terminal singularities and a canonical divisor that is relatively nef. We refer to [Mat02] for more information on the minimal model program.

The description of the Chow ring and the canonical divisor of the Bott-Samelson variety by Demazure in [Dem74] enables us to compute these objects for Schubert varieties and intermediate resolutions. Those are needed to use the minimal model program to classify all relative minimal models of cominuscule Schubert varieties.

Theorem 1.4. *Let $X_P(w)$ be a cominuscule Schubert variety. The relative minimal models of $X_P(w)$ birational to $X_P(w)$ are exactly the varieties $\widehat{X}(\widehat{w})$ obtained from a kind of neat ordering.*

We proof that existence and termination of flops holds for our varieties by a direct computation. Therefore, we do not need to use any recent progresses on the minimal model program. It can be directly checked that intermediate resolutions coming from a kind of neat, but not neat ordering, are not small. To summarize we get the following main theorem of this thesis.

Theorem 1.5. *Let $X_P(w)$ be a cominuscule Schubert variety over $\mathbb{C}$. Then the map $\widehat{\pi} : \widehat{X}(w) \to X_P(w)$ is a small resolution if and only if $\widehat{X}(w)$ is smooth and obtained from a neat ordering.*

2 Notation

Let G be a simply connected semisimple algebraic group over an algebraically closed field k. By T we denote a maximal torus in G and B is a Borel subgroup containing T. Furthermore, W shall be the Weyl group of G and R the set of all roots, while S is a set of simple roots corresponding to (B, T). We denote the set of positive roots by R^+, while R^- is the set of negative roots. Moreover, l is the length function on W corresponding to S. For each element $\alpha \in R$ we have the corresponding root subgroup $U_\alpha \subset G$. For a scheme X over k we write $A_*(X)$ for the Chow group. If X is a non singular variety (separated, integral and of finite type over k), $A^*(X)$ will denote the Chow ring.

3 Geometry of G/P

3.1 Schubert Varieties

We are going to introduce some specific parabolic subgroups. In order to do this, we need to recall some well known facts about parabolic subgroups and Schubert varieties. If P is a parabolic subgroup of G containing B, let $\Sigma_G(P)$ be the set of all simple roots α such that $U_{-\alpha} \nsubseteq P$. Most of the time we will simply write $\Sigma(P)$ if there's no chance for confusion.

Fact 3.1 ([Bou68]). *The map*

$$\Sigma : \{P \mid B \subseteq P \text{ is a subgroup of } G\} \to \mathcal{P}(S)$$

is a bijection.

Proof. This is just a reformulation of [Bou68] Theorem IV.§2.5.3. □

Notice, that we explicitly have the equality $\Sigma(P) = S \backslash \{\alpha \in S \mid Bs_\alpha B \subseteq P\}$.

The Weyl group of (P, T) is $W_P = \langle s_\alpha \mid \alpha \in S \backslash \Sigma(P) \rangle$. Let R_P (resp. R_P^+, R_P^-) denote the root subset of R (resp. R^+, R^-) spanned by the simple roots in $S \backslash \Sigma(P)$. For an element $w \in W$ we denote the Schubert variety $\overline{BwP}/P$ by $X_P(w)$. In view of the following fact W/W_P parametrizes the Schubert varieties in G/P.

Fact 3.2 (Bruhat decomposition modulo P). *The group G can be decomposed as the disjoint union*

$$G = \bigcup_{\overline{w} \in W/W_P} BwP.$$

Proof. This follows easily from [Bou68] Theorem IV.§2.5.3 and Lemma IV.§2.1.1. □

Definition 3.3. Let $w \in W$.

(i) We define $l_P(w)$ to be the length of some representative of minimal length of $\overline{w} \in W/W_P$.

(ii) Let $R_P(w)$ be the following set

$$R_P(w) := \{\beta \in R^+ |\ w^{-1}(\beta) \in R^- \backslash R_P^-\}.$$

Moreover, we define $N_P(w) = \# R_P(w)$.

The next statement is based on [LMS74] Proposition 1.4. It establishes the basic properties of $N_P(w)$.

Lemma 3.4. *Let $v, w \in W$ and $\alpha \in S$.*

(i) We have

$$N_P(s_\alpha w) = \begin{cases} N_P(w) - 1 & \text{,if } w^{-1}(\alpha) \in R^- \backslash R_P^- \\ N_P(w) + 1 & \text{,if } w^{-1}(\alpha) \in R^+ \backslash R_P^+ \\ N_P(w) & \text{,if } w^{-1}(\alpha) \in R_P \end{cases}.$$

(ii) The inequality $l_P(s_\alpha w) \geq l_P(w) - 1$ holds.

(iii) We have the equality $N_P(w) = l_P(w)$.

(iv) The inequality $l(ws_\alpha) < l(w)$ holds if and only if $w(\alpha) \in R^-$.

Proof. (i) Due to α being simple, the reflection s_α induces a bijection between R^+ and $(R^+ \cup \{-\alpha\}) \backslash \{\alpha\}$. Therefore, $R_P(w) \backslash \{\alpha\}$ is in bijection with (not necessary equal to) $R_P(s_\alpha w) \backslash \{\alpha\}$. Furthermore, we have

$$\alpha \in R_P(w) \Rightarrow \alpha \notin R_P(s_\alpha w).$$

The first part of the lemma follows now directly from the definition of $N_P(w)$.

(ii) Without loss of generality we may assume w to be a representative of minimal length modulo W_P. Assume that the inequality $l_P(s_\alpha w) < l(w) - 1$ holds. Then there is an element $x \in W_P$ such that $l(s_\alpha wx) < l(w) - 1$. That implies the inequalities

$$l(wx) \leq 1 + l(s_\alpha wx) < l(w).$$

That contradicts the minimality of w.

(iii) Notice, that $N_P(w)$ and $l_P(w)$ are well defined modulo W_P. Indeed, this follows directly from the definitions. Therefore, we may assume $l_P(w) = l(w)$ throughout the whole proof. That means w is a representative of minimal length of a class modulo W_P.

First of all we are going to prove $N_P(w) \leq l_P(w)$ by induction on $l_P(w)$. If the equality $l_P(w) = 0$ holds, then we get $w = 1$ and $N_P(w) = 0$. Let $w = s_{\alpha_1} \ldots s_{\alpha_n}$ be a reduced expression of w. That means $l_P(s_{\alpha_1} w) \leq l_P(w) - 1$. Using induction yields

$$N_P(w) \leq N_P(s_{\alpha_1} w) + 1 \leq l_P(s_{\alpha_1} w) + 1 \leq l_P(w).$$

Next we prove $N_P(w) \geq l_P(w)$ by induction on $N_P(w)$. For the case $N_P(w) = 0$ we will prove by induction on $N_B(w)$ that $w \in W_P$. Let the equality $N_B(w) = 0$ hold. It is known that the Weyl group acts simply transitive on the set of bases. Therefore, if $w \neq 1$ holds there would be a root $\beta \in S$ with $w^{-1}(\beta) \in R^-$. But due to $N_B(w) = 0$ we get $w = 1$. Assume $N_B(w) > 0$. Together with $N_P(w) = 0$ this shows the existence of a root $\beta \in S$ with $w^{-1}(\beta) \in R_P^-$. Therefore, using the first part of the lemma we get $N_B(s_\beta w) = N_B(w) - 1$ and $N_P(s_\beta w) = N_P(w) = 0$. By induction $s_\beta w \in W_P$. This implies $-\beta = (s_\beta w)w^{-1}(\beta) \in R_P$. Thus we get $w = s_\beta s_\beta w \in W_P$.

Now assume that $N_P(w) > 0$. Let $\beta \in S$ be a root with $w^{-1}(\beta) \in R^- \backslash R_P^-$. Then using the induction hypothesis and (ii) we get

$$\begin{aligned} N_P(w) &= N_P(s_\beta w) + 1 \\ &\geq l_P(s_\beta w) + 1 \\ &\geq l_P(w). \end{aligned}$$

(iv) We have

$$\begin{aligned} l(ws_\alpha) < l(w) &\Leftrightarrow l(s_\alpha w^{-1}) < l(w^{-1}) \\ &\Leftrightarrow N_B(s_\alpha w^{-1}) < N_B(w^{-1}) \\ &\Leftrightarrow w(\alpha) \in R^-. \end{aligned}$$

That concludes the proof. □

Now we want to make a connection to the dimensions of the Schubert varieties.

Lemma 3.5 ([LMS74], Prop. 1.3)**.** *Let $w \in W$. The Schubert cell BwP/P is isomorphic to $\prod_{\alpha \in R_P(w)} U_\alpha$ as varieties. In particular,* $\dim X_P(w) = N_P(w)$.

Proof. Let $e = P/P \in G/P$. Then BwP/P is the B-orbit of we. But this is also the U orbit of we, where U is the maximal unipotent subgroup in B. The stabilizer of we under the action of G is wPw^{-1}. Therefore, the stabilizer of we under the action of U is $U \cap wPw^{-1}$. This subgroup of U is generated by the one dimensional unipotent subgroups U_α, where

$$\alpha \in \{\beta \in R^+ |\ w^{-1}(\beta) \in R^+ \cup R_P^-\}.$$

This set is the complement of $R_P(w)$ in R^+. Thus, the lemma follows from the fact, that U is isomorphic to the product of the U_α for all $\alpha \in R^+$. □

Notice, that the isomorphism is given by multiplication with we. This will be of use in the next section. The lemma enables us to prove a very useful result about minimal length representatives modulo W_P. (see [LMS79] Lemma 1.0).

Proposition 3.6. *Let $w \in W$. Then the following are equivalent.*

(i) *w is a representative of minimal length modulo W_P*

(ii) $l(ww') = l(w) + l(w')$ *for all* $w' \in W_P$

(iii) $l(ws_\alpha) = l(w) + 1$ *for all* $\alpha \in S\backslash\Sigma(P)$

(iv) $w(\alpha) \in R^+$ *for all* $\alpha \in S\backslash\Sigma(P)$

(v) *the natural morphism* $p : G/B \to G/P$ *maps* $X_B(w)$ *birationally onto* $X_P(w)$

(vi) $\dim X_P(w) = \dim X_B(w)$

Proof. (i) $\Rightarrow$ (ii) We have the equalities

$$N_B(w) = l(w) = l_P(w) = N_P(w).$$

Thus, w^{-1} maps no positive root to R_P^-. The element w'^{-1} doesn't change the sign for roots in $R\backslash R_P$. Therefore, we get

$$l(ww') = N_B(ww') = N_B(w) + N_B(w') = l(w) + l(w').$$

That (ii) implies (iii) is trivial. Furthermore, (iii) implies (iv) due to Lemma 3.4 (iv). Moreover, (iv) implies $w(R_P^+) \subseteq R^+$, which is equivalent to $R_P^- \subseteq w^{-1}(R^-)$. Thus, the equality $R_P(w) = R_B(w)$ holds. In view of Lemma 3.5 we get (v). Again trivially (v) implies (vi). Finally from (vi) follows

$$l_P(w) = N_P(w) = \dim X_P(w) = \dim X_B(w) = l(w),$$

which is the statement of (i). □

Part (ii) of this proposition shows, that there is a unique representative of minimal length in every class of W/W_P.

We also have the following theorem proved for example in [BK05] Theorem 3.2.2.

Theorem 3.7. *All Schubert varieties are normal.*

The following theorem is due to [FMSS95] (see comments after Theorem 1).

Theorem 3.8. *The Chow group of a Schubert variety is the free abelian group generated by the Schubert subvarieties.*

3.2 Parabolic Subgroups

Having established these basic properties, we can introduce some concrete parabolic subgroups. They will be of importance in the construction of resolutions later on.

Definition 3.9. Let $w \in W$.

(i) We define P^w to be the parabolic subgroup of G given by

$$\Sigma(P^w) = \{\alpha \in S \mid l(ws_\alpha) < l(w)\}.$$

(ii) Let P_w be the stabilizer of the Schubert variety $X_{P^w}(w) = \overline{BwP^w}/P^w$.

For concrete calculations it will often be useful to have a description via simple roots of P_w. This and an intuitive idea of P^w is what we establish in the next proposition.

Proposition 3.10 ([Per07], p. 1259)**.** *For all $w \in W$ we have*

(i) $X_{P^w}(w) = \overline{P_w wP^w}/P^w$,

(ii) P^w *is the largest parabolic subgroup of* G *such that the map* $\overline{BwB}/B \to \overline{BwP^w}/P^w$ *is birational,*

(iii) $\Sigma(P_w) = \{\alpha \in S \mid \overline{s_\alpha w} > \overline{w}$ *for the Bruhat order in* $W/W_{P^w}\}$.

Proof. The first statement follows directly from the definition of P_w. The equivalence of (iii) and (v) in Proposition 3.6 shows (ii).

Let $\alpha \in S$. We always have $Bs_\alpha wP^w \subseteq Bs_\alpha BBwP^w \subseteq Bs_\alpha wP^w \cup BwP^w$. Therefore, we get $Bs_\alpha B \not\subseteq P_w$ if and only if $Bs_\alpha wP^w \not\subseteq \overline{BwP^w}$, which is equivalent to $\overline{s_\alpha w} > \overline{w}$. That proves (iii). □

Definition 3.11. Let $w \in W$.

(i) We denote by $\mathrm{Supp}(w)$ the set of all $\alpha \in S$ such that s_α appears in some reduced expression of w.

(ii) Let $\partial\,\mathrm{Supp}(w)$ be set of all $\alpha \in S \setminus \mathrm{Supp}(w)$ such that s_α does not commute with w.

(iii) We define G_w to be the smallest semisimple subgroup of G containing all the U_α for $\alpha \in \mathrm{Supp}(w)$.

The last proposition of this chapter shows that we may always assume the inclusion $X_P(w) \subseteq G_w/P$ for any Schubert variety. We will use this in further constructions to make them easier.

Proposition 3.12 ([Per07], p. 1259)**.** *Let $w \in W$.*

(i) Let $\widetilde{w}$ be a reduced expression of w. Then $\operatorname{Supp}(w)$ *is the set of all $\alpha \in S$ such that s_α appears in $\widetilde{w}$. That means the set* $\operatorname{Supp}(w)$ *only depends on one reduced decomposition of w.*

(ii) Let $P, Q \subseteq G$ be parabolic subgroups containing B. Furthermore, assume that $P \subseteq P_w$. Then we have an isomorphism $\overline{PwQ}/Q \cong \overline{(P \cap G_w)w(Q \cap G_w)}/(Q \cap G_w)$.

(iii) We have an isomorphism $X_{P^w}(w) \cong \overline{(P_w \cap G_w)w(P^w \cap G_w)}/(P^w \cap G_w)$.

Proof. Indeed, the relations in a Coxeter group do not change the support proving (i). To prove (ii) notice that $R_Q(w)$ is contained in the root system generated by the support of w. This implies

$$U_w := \prod_{\alpha \in R_Q(w)} U_\alpha \subseteq G_w.$$

In Lemma 3.5 we have shown $(PwQ)/Q \cong (U_w wQ)/Q$. We even have the following commutative diagram

$$\begin{array}{ccccccc}
G_w & & \hookrightarrow & & G & \hookleftarrow & U_w \\
\downarrow{\scriptstyle \cdot w} & & & & \downarrow{\scriptstyle \cdot w} & & \downarrow{\scriptstyle \cong} \\
G_w/(Q \cap G_w) & \xrightarrow{\cong} & (G_wQ)/Q & \hookrightarrow & G/Q & \hookleftarrow & (U_w wQ)/Q.
\end{array}$$

Indeed, the map $G_w/(Q \cap G_w) \to G/Q$ is injective. Since $G_w/(Q \cap G_w)$ is projective, the map is also closed. It is a closed immersion, simply because $G_w/(Q \cap G_w) \to (G_wQ)/Q$ is an isomorphism.

Using $U_w \subseteq G_w$ we get

$$\begin{aligned}
(PwQ)/Q &\cong (U_w wQ)/Q \\
&\cong (U_w w(Q \cap G_w))/(Q \cap G_w) \\
&\cong ((P \cap G_w)w(Q \cap G_w))/(Q \cap G_w).
\end{aligned}$$

Taking closures concludes the proof of (ii). The last statement is just a special case. □

3.3 Quivers Associated to a Reduced Expression

We associate a quiver to each reduced expression of an element in the Weyl group. In order to do this we need to introduce some further notation. The quiver will enable us to translate the combinatorial problems in the Weyl group into geometric questions on the quiver.

Let $\widetilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression of an element $w \in W$.

Definition 3.13. Let $i \in [1, r]$.

(i) We define (if it exists) the **successor** of i to be $s_{\widetilde{w}}(i) = \min\{j \in [i+1, r] \mid \beta_j = \beta_i\}$.

(ii) Similarly, $p_{\widetilde{w}}(i) = \max\{j \in [1, r-1] \mid \beta_j = \beta_i\}$ is the **predecessor** of i (also only if it exists).

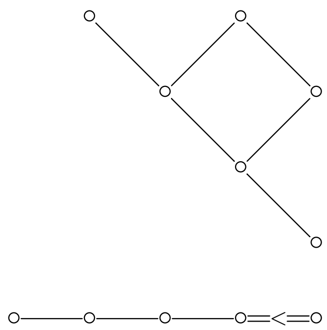
Note, that we will usually write $s(i)$ and $p(i)$ if there is no chance for confusion.

Definition 3.14. We define the quiver $Q_{\widetilde{w}}$ with vertices given by $[1, r]$. There is an arrow from i to j if $\langle \beta_i^\vee, \beta_j \rangle \neq 0$ and $i < j < s(i)$ (or $i < j$ if $s(i)$ does not exists). In addition, we give each vertex a color via the map $[1, r] \to S$ given by $i \mapsto \beta_i$.

The quiver uniquely determines the reduced expression up to commuting relations. Indeed, to get a reduced expression back from the quiver pick a vertex r, where no arrow starts. The rightmost element of the reduced expression will be s_{β_r}. Remove r from the quiver and proceed inductively, by multiplying the simple reflections from the left.

In general there may be multiple reduced expressions. Later on, we will deal with a special situation in which there is a unique reduced expression up to commuting relations for an element $\overline{w} \in W/W_P$. That opens the opportunity to get a lot of information about the Schubert variety out of the single quiver. Let us give an example.

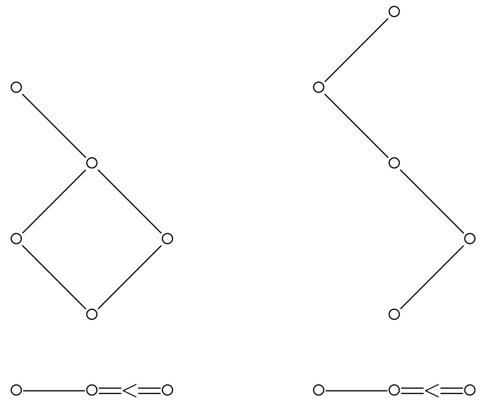
Example 3.15. Let G be the group Sp(10) which means $R = C_5$. We use the notation of [Bou68]. Now $w = s_{\alpha_2} s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} s_{\alpha_5}$ is a reduced expression modulo W_P for the parabolic subgroup P defined by $\Sigma(P) = \{\alpha_5\}$. This is annoying to check at the moment, so we will delay that. Later, it will be easy to verify. The corresponding quiver looks as follows, where the coloring is given by the projection of the vertices down to the quiver C_5.



As in this example we will always suppress the direction of the arrows.

It is not hard to check that this was the unique quiver for this element. In general, there are elements of the Weyl group with multiple quivers as the following example will show.

Example 3.16. The following two quivers correspond to the same element in the Weyl group corresponding to C_3. They are representatives of minimal length modulo W_P for $\Sigma(P) = \alpha_2$.



Proposition 3.17. *Let P be a maximal parabolic subgroup. Any quiver obtained from a reduced expression of a minimal length representative of an element in W/W_P is connected.*

Proof. Let w be a representative of minimal length of an element in W/W_P. We pick any reduced expression $\widetilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$. This implies $\Sigma(P) = \{\beta_r\}$. Assume that the quiver of $\widetilde{w}$ is not connected. Choose a maximal $i \in [1, r-1]$ which is not in the same connected component as β_r. Furthermore, let $j > i$ be minimal for the property $\langle \beta_j^\vee, \beta_i \rangle \neq 0$. If $\beta_i = \beta_j$, then s_{β_i} commutes with s_{β_k} for all $k \in [i, j]$ which contradicts the reducedness of w. If $\beta_i \neq \beta_j$, there is an arrow from i to j which contradicts the choice of i. Therefore, s_{β_i} commutes with s_{β_k} for all $k \in [i, r]$. But that is a contradiction to w being a representative of minimal length. □

4 The Bott-Samelson Variety

4.1 Definition

The Bott-Samelson variety is well known to desingularize Schubert varieties. In this section we are going to recall several of its definitions according to [Mag96] and [Dem74]. In the following $\widetilde{w} = (s_{\beta_1}, \dots, s_{\beta_r})$ will always be a reduced expression of an element $w \in W$.

Definition 4.1. Let P be any parabolic subgroup of G containing B and X any variety with a left B-action.

(i) We define a right B-action on $P \times X$ by $(p, x)b := (pb, b^{-1}x)$.

(ii) The induced quotient is denoted by $P \times^B X := (P \times X)/B$.

Lemma 4.2. *Let P be any parabolic subgroup of G containing B and X any variety with a left B-action. Then the morphism $P \times^B X \to P/B$ is a locally trivial fibration with fiber X and base P/B.*

Proof. Let w_0 be the unique maximal element in the Weyl group of P. Then Bw_0B/B is open in P/B. By Lemma 8.3.6 (ii) in [Spr98], there is a unipotent subgroup U_{w_0} such that we have an isomorphism $U_{w_0} \times B \to Bw_0B$ given by $(u, b) \mapsto uw_0b$. This yields a commutative diagram.

$$\begin{array}{ccccccc} Bw_0B/B \times X & \xrightarrow{\cong} & U_{w_0} \times X & \xrightarrow{\cong} & (U_{w_0} \times B) \times^B X & \xrightarrow{\cong} & Bw_0B \times^B X \\ & \searrow & & & & \swarrow & \\ & & Bw_0B/B & & & & \end{array}$$

Using the homogeneous P-action on P/B, we can cover it with open sets on all of which $P \times^B X$ becomes trivial. $\square$

Let Y be a P-variety containing X. Then we have a well defined proper map $P \times^B X \to Y$ given by $(p, x)B \mapsto px$. The next definition iterates this construction to get the Bott-Samelson variety.

Definition 4.3. Let P_{β_i} be the minimal parabolic subgroup containing B corresponding to β_i for $i \in [1, r]$. The **quotient Bott-Samelson variety** corresponding to $\widetilde{w}$ is given by

$$\widetilde{X}^{\mathrm{quo}}(\widetilde{w}) := P_{\beta_1} \times^B P_{\beta_2} \times^B \ldots \times^B P_{\beta_r}/B.$$

We get a B-action by acting on the leftmost factor. The multiplication map

$$\begin{aligned} \pi : \widetilde{X}^{\mathrm{quo}}(\widetilde{w}) &\to G/B \\ (p_1, \ldots, p_l) &\mapsto p_1 p_2 \ldots p_l B \end{aligned}$$

is called the **Bott-Samelson resolution**.

The following theorem justifies the name of π.

Theorem 4.4. *(i) The variety $\widetilde{X}^{\mathrm{quo}}(\widetilde{w})$ is smooth and projective.*

(ii) By restricting to the image $X_P(w)$ the map π becomes birational.

Proof. (i) By the previous lemma $\widetilde{X}^{\mathrm{quo}}(\widetilde{w})$ is defined via a tower of locally trivial $\mathbb{P}^1$-fibrations.
(ii) This is just a simple reformulation of Lemma 8.3.6 in [Spr98]. □

We will also denote the induced map $\widetilde{X}^{\mathrm{quo}}(\widetilde{w}) \to X_P(w)$ for any parabolic group P by π if the map $X_B(w) \to X_P(w)$ is birational. Next we define the Bott-Samelson variety as a subvariety of a product. Let $p_{\beta_i} : G/B \to G/P_{\beta_i}$ be the quotient morphism. It's fibers are isomorphic to $P_{\beta_i}/B \cong \mathbb{P}^1$. For any $x \in G/B$ we define $\mathbb{P}(x, \beta_i)$ to be $p_{\beta_i}^{-1}(p_{\beta_i}(x))$.

Definition 4.5. The **product Bott-Samelson variety** is given by

$$\widetilde{X}^{\mathrm{prod}}(\widetilde{w}) := \{(x_1, \ldots, x_r) \in (G/B)^r \mid x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r]\}.$$

We let B act diagonally on this variety. The map $\pi^{\mathrm{prod}} : \widetilde{X}^{\mathrm{prod}}(\widetilde{w}) \to X_B(w)$ is defined to be the restriction of the projection to the last factor.

The fact that π^{prod} really maps into $X_B(w)$ follows due to the next proposition. This definition is a reformulation of what Magyar calls the fiber product Bott-Samelson variety in [Mag96]. We are also going to restate his proof that the product and the quotient Bott-Samelson variety are isomorphic.

Proposition 4.6 ([Mag96], Theorem 1)**.** *The map*

$$\begin{aligned} \phi : \overbrace{G \times^B \ldots \times^B G}^{r \text{ times}}/B &\to (G/B)^r \\ (g_1, \ldots, g_r) &\mapsto (\overline{g_1}, \overline{g_1 g_2}, \ldots, \overline{g_1 \cdots g_r}) \end{aligned}$$

is a G-equivariant isomorphism. It restricts to an isomorphism of B-varieties

$$\phi' : \widetilde{X}^{\mathrm{quo}}(\widetilde{w}) \xrightarrow{\cong} \widetilde{X}^{\mathrm{prod}}(\widetilde{w}).$$

Moreover, $\pi^{\mathrm{prod}} \circ \phi' = \pi$.

Proof. Clearly, ϕ is well defined and G-equivariant. Let

$$\psi : (G/B)^r \to \overbrace{G \times^B \ldots \times^B G}^{r \text{ times}}/B$$
$$(\overline{h_1}, \ldots, \overline{h_r}) \mapsto \overline{(h_1, h_1^{-1}h_2, h_2^{-1}h_3, \ldots h_{r-1}^{-1}h_r)}.$$

It is easy to see that ψ is the inverse of ϕ. Let $(g_1, \ldots, g_r) \in \widetilde{X}^{\text{quo}}(\widetilde{w})$. Due to $g_i \in P_{\beta_i}$ we have $g_1 \cdots g_i P_{\beta_i} = g_1 \cdots g_{i-1} P_{\beta_i}$ for all $i \in [1, r]$. That implies $\overline{g_1 \cdots g_i} \in \mathbb{P}(\overline{g_1 \cdots g_{i-1}}, \beta_i)$. Therefore, ϕ' is well defined. For an element $(\overline{h_1}, \ldots, \overline{h_r}) \in \widetilde{X}^{\text{prod}}(\widetilde{w})$ we have the equality $h_i P_{\beta_i} = h_{i-1} P_{\beta_i}$ for all $i \in [2, r]$. That implies $h_{i-1}^{-1} h_i \in P_{\beta_i}$. Therefore, ψ restricts to a map $\psi' : \widetilde{X}^{\text{prod}}(\widetilde{w}) \to \widetilde{X}^{\text{quo}}(\widetilde{w})$. The first claim follows due to ψ' and ϕ' being inverse to each other.

The statement about π simply follows by the definition of ϕ. □

Because of this isomorphism, we are identifying the quotient and product Bott-Samelson variety, just call it the Bott-Samelson variety, and denote it by $\widetilde{X}(\widetilde{w})$.

There is another version of the Bott-Samelson variety similar to the product Bott-Samelson variety. Let $P^{\beta_i} := P^{s_{\beta_i}}$ be the maximal parabolic subgroup containing B corresponding to β_i for $i \in [1, r]$. For any $x \in G/B$ we denote the image of $\mathbb{P}(x_{i-1}, \beta_i)$ in G/P^{β_i} by $\overline{\mathbb{P}}(x_{i-1}, \beta_i)$.

Proposition 4.7 ([Per07], Remark 2.7)**.** *There is the following isomorphism of B-varieties.*

$$\widetilde{X}(\widetilde{w}) \cong \left\{ (x_1, \ldots, x_r) \in \prod_{i=1}^{r} G/P^{\beta_i} \mid x_0 = 1 \text{ and } x_i \in \overline{\mathbb{P}}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r] \right\}.$$

Proof. We regard the restriction of the projection $G/B \to G/P^{\beta_i}$ to $\mathbb{P}(x, \beta_i)$ for some $x \in G/B$. Because of $P^{\beta_i} \cap P_{\beta_i} = B$, this map is injective. Using Zariski's Main Theorem it is actually an embedding. □

In the next sections, we are going to focus on the case of $X_P(w)$, where $P = P^{\beta_r}$. In that case, the Bott-Samelson resolution is easily seen to be the projection on the last factor.

The last definition of the Bott-Samelson variety we are going to examine is a variation of the quotient Bott-Samelson variety. It was used by Demazure in [Dem74] to compute the Chow ring of $\widetilde{X}(\widetilde{w})$. We will recall his computations in the next sections.

There are $\beta_{r+1}, \ldots, \beta_N \in S$ for some $N \in \mathbb{N}$ such that $\widetilde{w_0} = (\beta_1, \ldots \beta_N)$ is a reduced expression of the unique longest element w_0 in W. We define a sequence of roots γ_i for $i \in [1, N]$.

$$\gamma_i := s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$$

Lemma 4.8 ([Bou68], p. 170, Corollary 2). *The γ_i are exactly the positive roots of R corresponding to S.*

Proof. It is well known that $\#R^+ = N$. The lemma follows if we can prove the inclusion $R^+ \subseteq \{\gamma_1, \ldots, \gamma_N\}$.

Let $\alpha \in R^+$. Due to $w_0^{-1}(\alpha) \in R^-$, there is an $i \in [1, n]$ such that $s_{\beta_{i-1}} \cdots s_{\beta_1}(\alpha) \in R^+$ and $s_{\beta_i} \cdots s_{\beta_1}(\alpha) \in R^-$. Since s_{β_i} permutes $R^+ \backslash \{\beta_i\}$, we get

$$s_{\beta_{i-1}} \cdots s_{\beta_1}(\alpha) = \beta_i.$$

That implies $\alpha = \gamma_i$. □

Note, that the equality $s_{\gamma_i} = s_{\beta_1} \cdots s_{\beta_{i-1}} s_{\beta_i} s_{\beta_{i-1}} \cdots s_{\beta_1}$ holds. We define $R_0 := R^+$ and $R_i := s_{\gamma_i}(R_{i-1})$ for $i \in [1, N]$.

Lemma 4.9. *For all $i \in [0, N]$ the set R_i is explicitly given as follows.*

$$R_i = \{-\gamma_1, \ldots, -\gamma_i, \gamma_{i+1}, \ldots, \gamma_N\}$$

This set consists of the positive roots with respect to the base $s_{\gamma_i} \cdots s_{\gamma_1}(S)$. Furthermore, for $i \neq 0$ the root γ_i is a simple root with respect to R_{i-1} and $-\gamma_i$ is a simple root with respect to R_i.

Proof. Since the equation $s_{\gamma_{i-1}} \cdots s_{\gamma_1}(\beta_i) = \gamma_i$ holds, we get the claim about simple roots. Because γ_i is a simple root with respect to R_{i-1}, the first statement follows by induction on i. □

Let B_i be the Borel subgroup corresponding to R_i and P_i the minimal parabolic subgroup corresponding to γ_i containing B_i and B_{i-1}.

Finally, we are able to formulate Demazure's definition of the Bott-Samelson variety. It is given by

$$\widetilde{X}^{\mathrm{Dem}}(\widetilde{w}) := P_1 \times^{B_1} P_2 \times^{B_2} \ldots \times^{B_{r-1}} P_r/B_r.$$

Proposition 4.10. *The Bott-Samelson variety $\widetilde{X}(\widetilde{w})$ is isomorphic to $\widetilde{X}^{\mathrm{Dem}}(\widetilde{w})$.*

Proof. For all $i \in [1, N]$ we get the following facts.

$$\begin{aligned} s_{\gamma_1} \cdots s_{\gamma_{i-1}} s_{\gamma_i} s_{\gamma_{i-1}} \cdots s_{\gamma_1} &= s_{\beta_i} \\ s_{\gamma_1} \cdots s_{\gamma_{i-1}}(R_{i-1} \cup R_i) &= R_0 \cup s_{\beta_i}(R_0) \\ s_{\gamma_1} \cdots s_{\gamma_{i-1}} P_i s_{\gamma_{i-1}} \cdots s_{\gamma_1} &= P_{\beta_i} \\ s_{\gamma_1} \cdots s_{\gamma_i} B_i s_{\gamma_i} \cdots s_{\gamma_1} &= B \\ P_i s_{\gamma_i} &= P_i \end{aligned}$$

Let n_i be a representative of s_{γ_i} and $v_i = n_1 \cdots n_i$. Putting the last three equations together we get

$$\begin{aligned}\widetilde{X}(\widetilde{w}) &= P_1 v_1 \times^{(v_1 B_1 v_1^{-1})} (v_1 P_2 v_2^{-1}) \times^{(v_2 B_2 v_2^{-1})} \ldots \times^{(v_{r-1} B_{r-1} v_{r-1}^{-1})} (v_{r-1} P_r v_r^{-1})/(v_r B v_r^{-1}) \\ &\cong P_1 \times^{B_1} P_2 \times^{B_2} \ldots \times^{B_{r-1}} P_r/B_r.\end{aligned}$$

The isomorphism is given by

$$(p_1, \ldots, p_r) \mapsto (p_1 v_1^{-1}, v_1^{-1} p_2 v_2, \ldots, v_{r-1}^{-1} p_r v_r).$$

□

4.2 Cycles on the Bott-Samelson Variety I

The structure of the Chow ring of the Bott-Samelson variety is well understood due to [Dem74]. In this section we are recalling Demazure's description of the Chow ring. For more information on intersection theory we refer to [Ful98].

Let X be a variety on which B operates faithfully from the right such that X/B exists, is smooth, projective and $p : X \to X/B$ is a locally trivial fibration with fiber B.

Definition 4.11. For each character $\lambda \in X^*(B)$ we define an invertible sheaf by

$$\mathcal{L}_X(\lambda) := \{\text{sections of } X \times^B k_\lambda \to X/B\}.$$

First of all, we need some basic properties of these sheaves.

Proposition 4.12. *Let $\lambda \in X^*(B)$.*

(i) For all open subsets $U \subseteq X/B$ we have the isomorphism

$$\Gamma(U, \mathcal{L}_X(\lambda)) \cong \{\varphi : p^{-1}(U) \to k \mid \varphi(xb) = \lambda(b)^{-1}\varphi(x) \text{ for all } x \in X, b \in B\} =: R.$$

(ii) The map $X^*(B) \to \operatorname{Pic}(X/B)$ defined by $\lambda \mapsto \mathcal{L}_X(\lambda)$ is a group homomorphism.

Proof. (i) For $s \in \Gamma(U, \mathcal{L}_X(\lambda))$ we define $\varphi_s : p^{-1}(U) \to k$ via $\varphi_s(x) = a$ if $s(\overline{x}) = \overline{(x, a)}$. By the definition of $X \times^B k$ the map φ_s is in R. If we have a map $\varphi \in R$, then $s_\varphi(\overline{x}) = \overline{(x, \varphi(x))}$ defines an element in $\Gamma(U, \mathcal{L}_X(\lambda))$.
(ii) Let $U \subseteq X/B$ be any open subset. Using the description of (i) we can define a morphism $\Gamma(U, \mathcal{L}_X(\lambda)) \otimes \Gamma(U, \mathcal{L}_X(\mu)) \to \Gamma(U, \mathcal{L}_X(\lambda + \mu))$ simply by multiplication for any $\lambda, \mu \in X^*(B)$. That induces a morphism of sheaves $\psi : \mathcal{L}_X(\lambda) \otimes \mathcal{L}_X(\mu) \to \mathcal{L}_X(\lambda+\mu)$. We can check locally that this is an isomorphism. Let $U \subseteq X/B$ be an open affine subset on which $X \to X/B$ becomes trivial. Then we have the isomorphism

$$\begin{aligned}\Gamma(U, \mathcal{L}_X(\lambda)) &= \{\varphi : U \times B \to k \mid \varphi(xb) = \lambda(b)^{-1}\varphi(x)\} \\ &\cong \{\varphi : U \to k\} = \Gamma(U, \mathcal{O}_U).\end{aligned}$$

The isomorphism is given by restricting to $U \times \{1\}$. Therefore, ψ is locally just the usual multiplication isomorphism $\mathcal{O}_U \otimes \mathcal{O}_U \to \mathcal{O}_U$. □

Taking the first Chern class of a line bundle defines a map $\mathrm{Pic}(X/B) \to A^1(X/B)$. Composing with the previous homomorphism induces a map $c_X : X^*(B) \to A^*(X/B)$.

In the case $X = G$ for G being some semisimple simply connected linear algebraic group, the map $\mathrm{Pic}(G/B) \to A^1(G/B)$ is actually an isomorphism. This is for example proven in [KKH89], Proposition 3.2. The Picard group of G/P, where P is a maximal parabolic subgroup containing B, can be computed as follows.

Proposition 4.13. *Let P be a maximal parabolic subgroup of G given by $\Sigma(P) = \{\alpha\}$ and ω the fundamental weight corresponding to $\alpha \in S$. The simple G-representation of highest weight ω is denoted by $V(\omega)$ and v is a highest weight vector. Then we have a closed embedding $G/P \hookrightarrow \mathbb{P}(V(\omega))$ given by $\overline{g} \mapsto g[v]$. Moreover, the ample generator $\mathcal{O}(1)$ of $\mathbb{P}(V(\omega))$ is pulled back to $\mathcal{L}_G(-\omega)$. Therefore G/P has an ample generator.*

Proof. For the first claim we need to show $\mathrm{Stab}_G([v]) = P$. Note, that the equality $\Sigma(P) = \{\beta \in S|\ \langle \beta^\vee, \omega\rangle \neq 0\}$ holds. Let n_β be a representative of the simple reflection s_β for some $\beta \in S$. Then $n_\beta v$ is of weight $s_\beta(\omega)$. The element n_β is in the group generated by U_β and $U_{-\beta}$. On the one hand, if $U_{-\beta}$ acts trivially on $[v]$, then n_β acts trivially on $[v]$ and $\langle \beta^\vee, \omega\rangle = 0$ holds. On the other hand, if $\langle \beta^\vee, \omega\rangle = 0$, then $U_{-\beta} = n_\beta^{-1} U_\beta n_\beta$ acts trivially on v.

The line bundle $\mathcal{O}(-1)$ of $\mathbb{P}(V(\omega))$ can be described as $\{([x], y)|\ x \in V(\omega), y \in kx\}$. Therefore, we need to find an isomorphism

$$G \times^P k_\omega \xrightarrow{\cong} \{(g[v], y)|\ g \in G, y \in kgv\}.$$

But this is just given by $\overline{(g, \lambda)} \mapsto (g[v], \lambda g v)$. □

Let P be a semi direct product of a unipotent connected group U and a reductive connected group L of semisimple rank 1. Note, that any minimal parabolic subgroup P of G is of this form. Moreover, let T be a maximal torus of L, α one of the two roots of L with respect to T and $U_\alpha, U_{-\alpha}$ the two corresponding unipotent subgroups. Then $B = TU_\alpha U$ and $B' = TU_{-\alpha}U$ are the two Borel subgroups of P containing T. We define another space $X' := X \times^B P$. Let $f : X'/B' \to X/B$ be the map induced by the first projection. First of all, we need to check that X' fulfills the same properties as X.

Proposition 4.14. *The space X'/B' exists as a variety, is smooth, projective and f is a locally trivial fibration with fiber P/B'.*

Proof. Let us first assume that X'/B' is a well defined variety. Furthermore, U shall be an open subset of X/B such that $p^{-1}(U) \cong U \times B$. Thus, we get the isomorphism

$f^{-1}(U) \cong U \times F \times^B P/B' \cong U \times P/B'$. Therefore, f is a locally trivial fibration implying smoothness. To get the existence we can define X'/B' via this open covering. Finally, the variety X'/B' is projective because $P/B \cong \mathbb{P}^1$. □

This setup actually describes the situation of one step in the last definition of the Bott-Samelson variety in Proposition 4.10. Studying the map f will allow us to use induction.

Proposition 4.15. *Let $n \in L$ be a representative of the only non trivial element s_α of the Weyl group. Then $\sigma : X/B \to X'/B'$ defined by $\overline{x} \mapsto \overline{(x, n)}$ is a well defined section of f.*

Proof. This follows from $n^{-1}Bn = B'$. □

At this point we establish a connection between f, the section σ and the locally free sheaves defined before.

Lemma 4.16. *We have the isomorphism $\sigma^*(\mathcal{L}_{X'}(\lambda)) \cong \mathcal{L}_X(s_\alpha(\lambda))$ for all characters $\lambda \in X^*(T)$.*

Proof. To prove this we regard the locally free sheaves as line bundles. We have the following pullback diagram.

$$\begin{array}{ccc} X \times^B BnB' \times^{B'} k_\lambda & \hookrightarrow & X \times^B P \times^{B'} k_\lambda \\ \downarrow & & \downarrow \\ X/B & \xrightarrow{\ \sigma\ } & X \times^B P/B' \end{array}$$

Every class in $X \times^B BnB' \times^{B'} k_\lambda$ has a representative of the form (x, n, a). Therefore, it is isomorphic to the quotient of $X \times \{n\} \times k_\lambda$ given by the identifications $(x, n, a) \sim (xb, b^{-1}nb', \lambda(b')^{-1}a)$ for $b \in B$, $b' \in B'$, $x \in X$, $a \in k_\lambda$ and $b^{-1}nb' = n$. But that is the same as $(x, n, a) \sim (xb, n, \lambda(n^{-1}bn)^{-1}a)$. The lemma follows from $\lambda(n^{-1}bn) = s_\alpha(\lambda)(b)$. □

Lemma 4.17. *The degree of $\mathcal{L}_{X'}(\lambda)$ along every fiber of f is given by $\langle \alpha^\vee, \lambda \rangle$ for all $\lambda \in X^*(T)$.*

Proof. We choose an $x \in X$ and replace X by xB. Thus, we may assume X/B to be trivial. Furthermore, since B acts faithfully on X, we get $X \times^B P/B' \cong P/B'$. Therefore, we get the isomorphism

$$\mathcal{L}_{X'}(\lambda) \cong \{\text{sections of } P \times^{B'} k_\lambda \to P/B'\}.$$

The commutator subgroup $L' = D(L)$ is a semisimple group that has a maximal torus $T' \subseteq T$. We have the isomorphisms $L'/T'U_{-\alpha} \cong P/B'$ and $L' \times^{T'U_{-\alpha}} k_\lambda \cong P \times^{B'} k_\lambda$. The group L' is of rank 1 which means it is either SL_2 or PSL_2. Since the quotients do not differ for both these groups, we may assume $L' = \mathrm{SL}_2$. The only difference to the situation $L = \mathrm{SL}_2$ is that we have a bigger torus with more characters. However, if $\langle \alpha^\vee, \lambda \rangle = 0$, the action of $T'U_{-\alpha}$ on k_λ is trivial. Therefore, the lemma is true in that situation. Since the map $X^*(T) \to \mathrm{Pic}(P/B')$ given by $\lambda \mapsto \mathcal{L}_{X'}(\lambda)$ is a homomorphism, we only need to look at the unique character λ of T' with $\langle \alpha^\vee, \lambda \rangle = 1$. The quotient P/B' is isomorphic to $\mathbb{P}^1$. Therefore, it suffices to check that the space of global section of $\mathcal{L}_{X'}(\lambda)$ is of dimension 2. This is done via a concrete calculation in SL_2 as follows.

First of all, using Proposition 4.12 the global sections of $\mathcal{L}_{X'}(\lambda)$ are given by

$$\{\varphi : L \to k \mid \varphi(ltu) = \lambda(t)^{-1}\varphi(l) \ \forall l \in L, t \in T', u \in U_{-\alpha}\}.$$

Due to the Bruhat decomposition, there is an open covering $L = U_{-\alpha}TU_\alpha \cup nU_{-\alpha}TU_\alpha$. One can compute the dimension explicitly as follows. First define two functions on $U_{-\alpha}$ and $nU_{-\alpha}$ that extend in only one possible way to the open sets in the covering. Then one can compute the condition of them coinciding on the intersection via an explicit calculation with matrices in SL_2. We will leave this calculation to the reader. $\square$

By D we denote the divisor given by the image of the section σ. The corresponding locally free sheaf of rank 1 will be denoted by $\mathcal{L}(D)$.

Lemma 4.18. *The sheaf $\sigma^*(\mathcal{L}(D))$ is isomorphic to $\mathcal{L}_X(-\alpha)$.*

Proof. Let $n \in L$ be a representative of the only non trivial element s_α of the Weyl group. There is a morphism $\widetilde{\sigma} : X \to X \times P/B'$ given by $x \mapsto (x, \overline{n})$. Furthermore, we define $\widetilde{D} := X \times \{\overline{n}\}$. The following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{\sigma}} & X \times P/B' \\ \downarrow & & \downarrow \\ X/B & \xrightarrow{\sigma} & X \times^B P/B' \end{array}$$

Therefore, we have an isomorphism $\sigma^*(\mathcal{L}(D)) \cong \widetilde{\sigma}^*(\mathcal{L}(\widetilde{D}))/B$. To get $\widetilde{\sigma}^*(\mathcal{L}(\widetilde{D}))$ we need to compute $\mathcal{L}(\widetilde{D})_{|\widetilde{D}}$. We also have the following commutative diagram.

$$\begin{array}{ccc} X \times P/B' & \hookleftarrow & \widetilde{D} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ P/B' & \hookleftarrow & \{\overline{n}\}. \end{array}$$

Thus, we get the isomorphism $\mathcal{L}(\widetilde{D})_{|\widetilde{D}} \cong (p^*\mathcal{L}(\{\overline{n}\}))_{|\widetilde{D}} \cong q^*(\mathcal{L}(\{\overline{n}\})_{|\{\overline{n}\}})$. It suffices to check that B acts via $-\alpha$ on $\mathcal{L}(\{\overline{n}\})_{|\{\overline{n}\}}$. This can be done via a concrete calculation in SL_2.

Let B be the Borel subgroup of upper triangular matrices in SL_2. Due to the isomorphism $P/B' \cong \mathbb{P}^1 = \mathrm{Proj}(k[x,y])$, we can identify n with $[1:0]$. With this identification we have the equality $\mathcal{L}(\{\overline{n}\})_{|\{\overline{n}\}} = (x/y \cdot k[y/x])/(k[y,x])$. Indeed, B acts via $-\alpha$ on it. $\square$

Lemma 4.19. *The sheaf* $\Omega_{(X'/B')/(X/B)}$ *is isomorphic to* $\mathcal{L}_{X'}(-\alpha)$.

Proof. We can do this computation again in SL_2. Let L be the line bundle given by $X \times^B k^2 \to X/B$. Then $X \times^B P/B'$ is the projective bundle $\mathbb{P}(L)$. By [Har77] Exercise 8.4(b) there is an isomorphism $\Omega_{(X\times^B P/B')/(X/B)} \cong (f^* \bigwedge^2 L) \otimes \mathcal{O}_{X\times^B P/B'}(-2)$. On P/B' we can use Proposition 4.13 to get $\mathcal{L}_P(\omega) \cong \mathcal{O}_{P/B'}(1)$. That implies the isomorphism $\mathcal{O}_{X\times^B P/B'}(-2) \cong \mathcal{L}_{X\times^B P}(-\alpha)$. Furthermore, for all $\lambda, \mu \in k$ and $b \in B$ we have $b \cdot \lambda \wedge \mu = \lambda \wedge \mu$. Therefore, $\bigwedge^2 L$ is trivial concluding the proof. □

Lemma 4.20. *For all characters* $\lambda \in X^*(T)$ *we get an isomorphism*

$$\mathcal{L}_{X'}(\lambda) \cong f^*(\mathcal{L}_X(\lambda)) \otimes \mathcal{L}(D)^{\otimes\langle\alpha^\vee,\lambda\rangle}.$$

Proof. Since $P/B' \cong \mathbb{P}^1$, the locally trivial fibration $f : X'/B' \to X/B$ is a projective bundle of rank 1. Therefore, we get $\mathrm{Pic}(X'/B') \cong \mathrm{Pic}(X/B) \oplus \mathbb{Z}\mathcal{O}(1)$. The second projection $\mathrm{Pic}(X'/B') \to \mathbb{Z} \cdot \mathcal{O}(1)$ is given by restriction to the fiber.

According to Lemma 4.17 the degrees of $\mathcal{L}_{X'}(\lambda)$ and $\mathcal{L}(D)^{\otimes\langle\alpha^\vee,\lambda\rangle}$ are the same along every fiber of f. Hence, there is a sheaf $\mathcal{L} \in \mathrm{Pic}(X/B)$ such that

$$f^*(\mathcal{L}) \cong \mathcal{L}_{X'}(\lambda) \otimes \mathcal{L}(D)^{\otimes(-\langle\alpha^\vee,\lambda\rangle)}.$$

Finally, we can conclude the proof by the following isomorphism using Lemma 4.16 and 4.18

$$\begin{aligned}\mathcal{L} &\cong \sigma^* f^* \mathcal{L} \\ &\cong \sigma^*\mathcal{L}_{X'}(\lambda) \otimes \sigma^*\mathcal{L}(D)^{\otimes(-\langle\alpha^\vee,\lambda\rangle)} \\ &\cong \mathcal{L}_X(s_\alpha(\lambda) + \langle\alpha^\vee,\lambda\rangle\alpha) \\ &= \mathcal{L}_X(\lambda).\end{aligned}$$ □

Finally, we can describe $A^*(X'/B')$ in terms of $A^*(X/B)$. We denote the corresponding class of D by ξ. Obviously, we have the equality $\xi = \sigma_*(1) = c_1(\mathcal{L}(D))$, where c_1 denotes the first Chern class.

Proposition 4.21. *The ring homomorphism* $f^* : A^*(X/B) \to A^*(X'/B')$ *is injective. It identifies* $A^*(X'/B')$ *with* $A^*(X/B)[\xi]/(\xi^2 + c_X(\alpha)\xi)$.

Proof. We have the equality $\sigma^* \circ f^* = \mathrm{id}$ proving the injectivity of f^*. Next, we will prove the relation $\xi^2 + c_X(\alpha)\xi = 0$. The projection formula ([Ful98] Proposition 8.3) yields the equality $\sigma_*\sigma^*(a') = \xi a'$ for all $a' \in A^*(X'/B')$. For any element $a \in A^*(X/B)$ we set $a' = f^*(a)$ to get $\sigma_*(a) = \xi f^*(a)$. Choosing $a' = \xi$ and $a = \sigma^*(a')$ the two last equations yield $\xi^2 = \sigma_*\sigma^*(\xi) = \xi f^*\sigma^*(\xi)$. Because taking Chern classes and pulling back commutes ([Ful98] Theorem 3.2), we can use Lemma 4.18 to get $\sigma^*(\xi) = c_X(-\alpha) = -c_X(\alpha)$. This proves the relation.

Using well known facts about projective bundles ([Ful98] Theorem 3.3) we know that $A^*(X'/B')$ is a free $A^*(X/B)$ module with basis $(1, \mathcal{O}(1))$. We are left to prove, that $(1, \xi)$ is also a basis. This is true because the degree of ξ along the fibers of f is 1. $\square$

Despite this result, we still lack a description of $c_X(\alpha)$. However, the next proposition gives a description of $c_{X'}$ in terms of c_X.

Proposition 4.22. *For all $\lambda \in X^*(T)$ we get the equality $c_{X'}(\lambda) = c_X(\lambda) + \langle \alpha^\vee, \lambda \rangle \xi$.*

Proof. Applying c_1 to the equality in Lemma 4.20 immediately shows the statement. $\square$

4.3 Cycles on the Bott-Samelson Variety II

In this section we are going to give a description of the Chow ring of the Bott-Samelson variety. Moreover, we will determine all ample divisors. This is needed for our application of Mori theory in the later sections. Through all of this we will use the notation of section 4.1. Furthermore, we define $X_0 = B_0$ and $X_i = X_{i-1} \times^{B_{i-1}} P_i$ for all $i \in [1, r]$. Using the results of the last section there are sections $\sigma_i : X_{i-1}/B_{i-1} \to X_i/B_i$ of the maps $f_i : X_i/B_i \to X_{i-1}/B_{i-1}$. Moreover, we define $\xi_i := f_r^* \cdots f_{i+1}^*(\sigma_i)_*(1)$.

Theorem 4.23 ([Dem74], Proposition 4.1). *(i) For all $i, j \in [1, r]$ there is the equality*

$$c_{X_i}(\gamma_j) = \sum_{l=1}^{i} \langle \gamma_l^\vee, \gamma_j \rangle \xi_l.$$

(ii) For all $i \in [0, r]$, there is an isomorphism

$$A^*(X_i/B_i) \cong \mathbb{Z}[\xi_1, \ldots, \xi_i]/(\xi_j^2 + c_{X_{j-1}}(\gamma_j)\xi_j \mid j \in [1, i]).$$

Proof. The first statement follows by an induction on i from Proposition 4.22, while the second follows from Proposition 4.21 also using induction on i. $\square$

The divisors $Z_i := f_r^{-1} \cdots f_{i+1}^{-1}(\sigma(X_{i-1}/B_{i-1}))$ are representatives of ξ_i for all $i \in [1, r]$. The Z_i can be determined for the other descriptions of the Bott-Samelson variety as follows.

Proposition 4.24. *(i) In the quotient Bott-Samelson variety, we have*

$$Z_i = \{(x_1, \ldots, x_r) \in \widetilde{X}^{quo}(\widetilde{w}) \mid x_i = 1\}.$$

(ii) Regarding the product Bott-Samelson variety, we get

$$Z_i = \{(x_1, \ldots, x_r) \in \widetilde{X}^{prod}(\widetilde{w}) \mid x_i = x_{i-1}\}.$$

(iii) With the description of Proposition 4.7, Z_i is described as

$$Z_i = \{(x_1, \ldots, x_r) \in \widetilde{X}(\widetilde{w}) \mid x_i = x_{p(i)}\}.$$

Proof. Parts (i) and (ii) follow directly from the isomorphism between the different definitions of the Bott-Samelson variety. To get part (iii), we will use part (ii). We need to prove that the equivalence $g_iB = g_{i-1}B \Leftrightarrow g_iP^{\beta_i} = g_{p(i)}P^{\beta_i}$ holds for any $(\overline{g_1}, \ldots, \overline{g_r}) \in \widetilde{X}^{\text{prod}}(\widetilde{w})$.

If $g_iB = g_{i-1}B$, then we have

$$g_{p(i)} \in g_{i-1}P_{\beta_{i-1}} \cdots P_{\beta_{p(i)-1}} = g_iP_{\beta_{i-1}} \cdots P_{\beta_{p(i)-1}} \subseteq g_iP^{\beta_i}.$$

However, if $g_iP^{\beta_i} = g_{p(i)}P^{\beta_i}$ holds, then we get

$$g_{i-1} \in g_{p(i)}P_{\beta_{p(i)+1}} \cdots P_{\beta_{i-1}} \subseteq g_{p(i)}P^{\beta_i} = g_iP^{\beta_i}$$

and $g_{i-1} \in g_iP_{\beta_i}$. Due to $P_{\beta_i} \cap P^{\beta_i} = B$, the equality $g_iB = g_{i-1}B$ follows. □

There is even the following result due to [LT04] Proposition 3.5.

Proposition 4.25. *The divisors $\xi_1, \ldots \xi_r$ form a basis of the monoid of effective divisors.*

Proof. Let $[D] = \sum_{i=0}^{r} m_i\xi_i$ be any effective divisor. We need to prove $m_i \geq 0$ for all $i \in [1, r]$. Since all Z_i are B-stable, the global sections of $\mathcal{L}(D)$ form a finite dimensional B-representation. Using Borel's fixed point theorem we get a global section s fixed under B up to scalar multiplication. Therefore, the zero divisor $V(s)$ of s is a B-stable divisor of $\widetilde{X}(\widetilde{w})$. Since $\widetilde{X}(\widetilde{w})\setminus\bigcup_{i=0}^{r}\xi_i$ is isomorphic to BwB/B, the set $\widetilde{X}(\widetilde{w})\setminus\bigcup_{i=0}^{r}\xi_i$ is a dense B-orbit. That implies $V(s) \subseteq \bigcup_{i=0}^{r}\xi_i$. Therefore, there are $m_i^{'} \geq 0$ for all $i \in [1, r]$ such that $[D] = [V(s)] = \sum_{i=1}^{r} m_i^{'}\xi_i$. Because $(\xi_i)_{i\in[1,r]}$ is a base of $A^1(\widetilde{X})(\widetilde{w})$, we get $m_i = m_i^{'} \geq 0$. □

For any set $K \subset [1, r]$ we define Z_K to be $\bigcap_{i\in K} Z_i$. The variety Z_K has codimension $\#K$. Indeed, this can be seen by $Z_K = \{(p_1, \ldots, p_r) \in \widetilde{X}^{\text{quo}}(\widetilde{w}) | \ p_i = 1 \text{ for all } i \in K\}$. Using this description of Z_K also enables us to compute the image of Z_K in the Schubert variety $X_P(w)$.

Proposition 4.26. *The image $\pi(Z_K)$ is equal to the Schubert variety $X_P(v)$, where v is the longest element that can be written as a subword of $\widetilde{w}$ without any s_{β_i} for all $i \in K$.*

Proof. This follows from [Bou68] Section IV.§2 equation (3) and Theorem IV.§2.4.2. □

By T_i we denote the pullback via $f_r^* \cdots f_{i+1}^*$ of the relative tangent sheaf of f_i. The curve $Z_{[1,r]\setminus\{i\}}$ will be denoted by C_i. For notational simplicity we identify any line bundle $\mathcal{L} \in \operatorname{Pic}(\widetilde{X}(\widetilde{w}))$ with its Chern class $c_1(\mathcal{L}) \in A^1(\widetilde{X}(\widetilde{w}))$.

In the following, we are recalling some equations from [Per05].

Proposition 4.27 ([Per05], Proposition 3.3)**.** *The following equality holds in $A^*(\widetilde{X}(\widetilde{w}))$ for all $i, j \in [1, r]$.*

$$[C_i] \cdot \xi_j = \begin{cases} 0 & , \textit{ if } i > j \\ 1 & , \textit{ if } i = j \\ \langle \beta_i^\vee, \beta_j \rangle & , \textit{ if } i < j \end{cases}$$

Proof. We have the equality $C_i = \prod_{k \neq i} \xi_k$. Because $Z_{[1,r]}$ is just a point, we have $[C_i] \cdot \xi_i = 1$. An easy induction on j yields

$$[C_i] \cdot \xi_j = \begin{cases} 0 & , \text{ if } i > j \\ 1 & , \text{ if } i = j \\ \sum_{k=1}^{j-i} (-1)^k \sum_{i=i_0<\ldots<i_k=j} \prod_{x=0}^{k-1} \langle \gamma_{i_x}^\vee, \gamma_{i_{x+1}} \rangle & , \text{ if } i < j. \end{cases}$$

Therefore, we are left to prove

$$\sum_{k=1}^{j-i} (-1)^k \sum_{i=i_0<\ldots<i_k=j} \prod_{x=0}^{k-1} \langle \gamma_{i_x}^\vee, \gamma_{i_{x+1}} \rangle = \langle \beta_i^\vee, \beta_j \rangle.$$

We have $\beta_i = s_{\gamma_1} \cdots s_{\gamma_{i-1}}(\gamma_i)$. Furthermore, the equalities

$$\begin{aligned} \langle \beta_i^\vee, \beta_j \rangle &= \langle s_{\gamma_1} \cdots s_{\gamma_{i-1}}(\gamma_i), s_{\gamma_1} \cdots s_{\gamma_{j-1}}(\gamma_j) \rangle \\ &= \langle \gamma_i^\vee, s_{\gamma_i} \cdots s_{\gamma_{j-1}}(\gamma_j) \rangle \\ &= -\langle \gamma_i^\vee, s_{\gamma_{i+1}} \cdots s_{\gamma_{j-1}}(\gamma_j) \rangle \end{aligned}$$

hold. There are $x_{k,j} \in \mathbb{Z}$ such that $s_{\gamma_i} \cdots s_{\gamma_{j-1}}(\gamma_j) = \sum_{k=i}^{j} x_{k,j} \gamma_k$. Again doing an induction on j yields

$$x_{i,j} = \sum_{k=1}^{j-i} (-1)^k \sum_{i=i_0<\ldots<i_k=j} \prod_{x=0}^{k-1} \langle \gamma_{i_x}^\vee, \gamma_{i_{x+1}} \rangle.$$

Moreover, we get the following two equalities

$$\langle \gamma_i^\vee, s_{\gamma_i} \cdots s_{\gamma_{j-1}}(\gamma_j) \rangle = \sum_{k=i}^{j} x_{k,j} \langle \gamma_i^\vee, \gamma_k \rangle,$$

$$-\langle \gamma_i^\vee, s_{\gamma_{i+1}} \cdots s_{\gamma_{j-1}}(\gamma_j) \rangle = -\sum_{k=i+1}^{j} x_{k,j} \langle \gamma_i^\vee, \gamma_k \rangle.$$

Adding these equations yields $2\langle \gamma_i^\vee, s_{\gamma_i} \cdots s_{\gamma_{j-1}}(\gamma_j) \rangle = x_{i,j} \langle \gamma_i^\vee, \gamma_i \rangle = 2x_{i,j}$. That concludes the proof. □

The next goal is to describe T_i in terms of $\xi_1, \dots, \xi_r$.

Proposition 4.28 ([Per05], Fact 3.7 & Corollary 3.8)**.**

(i) For all characters $\lambda \in X^*(T)$ *we have* $\mathcal{L}_{X_i}(\lambda) = \sum_{k=1}^{i} \langle \gamma_k^\vee, \lambda \rangle \xi_k$.

(ii) The equality $T_i = \sum_{k=1}^{i} \langle \gamma_k^\vee, \gamma_i \rangle \xi_k$ *holds for all* $i \in [1, r]$.

Proof. The first part is a direct consequence of Lemma 4.20. Together with Lemma 4.19 we get the second part. □

Since we defined $\widetilde{X}(\widetilde{w})$ via a sequence of $\mathbb{P}^1$-fibrations, we can compute the canonical divisor as $-K_{\widetilde{X}(\widetilde{w})} = \sum_{i=1}^{r} T_i$. The previous proposition yields

$$\begin{aligned} -K_{\widetilde{X}(\widetilde{w})} &= \sum_{i=1}^{r} \sum_{k=1}^{i} \langle \gamma_k^\vee, \gamma_i \rangle \xi_k \\ &= \sum_{k=1}^{r} \left(\sum_{i=k}^{r} \langle \gamma_k^\vee, \gamma_i \rangle \right) \xi_k. \end{aligned}$$

Proposition 4.29 ([Per05], Proposition 3.11)**.** *For all* $i, j \in [1, r]$ *we have the equation*

$$[C_i] \cdot T_j = \begin{cases} 0 & \text{, if } i > j \\ \langle \beta_i^\vee, \beta_j \rangle & \text{, if } i \leq j. \end{cases}$$

Proof. Due to the last two propositions, the case $i > j$ is clear. For $i = j$ we get $[C_i] \cdot T_i = \sum_{k=1}^{i} \langle \gamma_k^\vee, \gamma_i \rangle [C_i] \xi_k = 2$. In the case $i < j$ we have the equalities

$$\begin{aligned} [C_i] \cdot T_j &= \sum_{k=1}^{j} \langle \gamma_k^\vee, \gamma_j \rangle [C_i] \xi_k \\ &= \langle \gamma_i^\vee, \gamma_j \rangle + \sum_{k=i+1}^{j-1} \langle \beta_i^\vee, \beta_k \rangle \langle \gamma_k^\vee, \gamma_j \rangle + 2 \langle \beta_i^\vee, \beta_j \rangle. \end{aligned}$$

Therefore, we are left to prove the next lemma. □

Lemma 4.30 ([Per05], Proposition 3.13)**.** *For all* $i, j \in [1, r]$ *with* $i < j$ *we have the formula*

$$\sum_{k=i+1}^{j-1} \langle \beta_i^\vee, \beta_k \rangle \langle \gamma_k^\vee, \gamma_j \rangle = -\langle \gamma_i^\vee, \gamma_j \rangle - \langle \beta_i^\vee, \beta_j \rangle.$$

Proof. As in the proof Proposition 4.27 we get

$$\langle \gamma_k^\vee, \gamma_j \rangle = \sum_{u=1}^{j-k} (-1)^u \sum_{k=i_0 < \dots < i_u = j} \prod_{x=0}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle$$

by exchanging the roles of γ_i and β_i. Setting $i_{-1} = i$, we can prove the proposition by the following series of equalities.

$$\begin{aligned}
\sum_{k=i+1}^{j-1} \langle \gamma_k^\vee, \gamma_j \rangle \langle \beta_i^\vee, \beta_k \rangle &= \sum_{k=i+1}^{j-1} \sum_{u=1}^{j-k} (-1)^u \sum_{k=i_0<\ldots<i_u=j} \langle \beta_i^\vee, \beta_k \rangle \prod_{x=0}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle \\
&= \sum_{u=1}^{j-i-1} (-1)^u \sum_{k=i+1}^{j-u} \sum_{k=i_0<\ldots<i_u=j} \prod_{x=-1}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle \\
&= \sum_{u=1}^{j-i-1} (-1)^u \sum_{i=i_{-1}<\ldots<i_u=j} \prod_{x=-1}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle \\
&= \sum_{u=2}^{j-i} (-1)^{u-1} \sum_{i=i_0<\ldots<i_u=j} \prod_{x=0}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle \\
&= -\langle \beta_i^\vee, \beta_j \rangle - \sum_{u=1}^{j-i} (-1)^u \sum_{i=i_0<\ldots<i_u=j} \prod_{x=0}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle \\
&= -\langle \beta_i^\vee, \beta_j \rangle - \langle \gamma_i^\vee, \gamma_j \rangle
\end{aligned}$$

□

Having a good description of the Chow ring we are now examining the ample divisors on $\widetilde{X}(\widetilde{w})$. Furthermore, the cone of effective curves will be described. This is also called the Mori cone (see [Mat02] for further information on this topic). For the rest of this section we will only use the Bott-Samelson variety as described in Proposition 4.7. There are morphisms $p_i : \widetilde{X}(\widetilde{w}) \to G/P^{\beta_i}$ induced by the projections of $\prod_{i=1}^r G/P^{\beta_i}$ for all $i \in [1, r]$. Since P^{β_i} is a maximal parabolic subgroup, the Picard group of G/P^{β_i} is isomorphic to $\mathbb{Z}$ and generated by a very ample invertible sheaf $\mathcal{O}(1)$. We define $\mathcal{L}_i$ to be the pullback $p_i^*\mathcal{O}(1)$. Furthermore, we define

$$Y_i := \{(x_1, \ldots, x_r) \in \prod_{j=1}^r G/P^{\beta_j} \mid x_j = \overline{1} \text{ for all } j \neq i \text{ and } x_i \in \mathbb{P}(\overline{1}, \beta_i)\}.$$

Our goal is to proof that the $\mathcal{L}_i$ form a base of the closure of the cone of ample divisors, while the Y_i form the dual base of the cone of effective curves in $\widetilde{X}(\widetilde{w})$. At first, we need to prove that Y_i is a curve in the Bott-Samelson variety.

Lemma 4.31 ([Per07], Lemma 2.12)**.** *We have the inclusion $Y_i \subset \widetilde{X}(\widetilde{w})$ for all $i \in [1, r]$.*

Proof. We only need to prove $\overline{1} \in \overline{\mathbb{P}}(\overline{g}, \beta_j)$ for all $\overline{g} \in \mathbb{P}(\overline{1}, \beta_i)$ and $j \in [i + 1, s(i) - 1]$. Since $\overline{g} \in \mathbb{P}(\overline{1}, \beta_i)$, we get $g \in P_{\beta_i}$. By definition we have $\overline{g} \in \mathbb{P}(\overline{g}, \beta_j)$. The inclusion $P_{\beta_i} \subseteq P^{\beta_j}$ implies $\overline{g} = \overline{1} \in G/P^{\beta_j}$ concluding the proof. □

Proposition 4.32 ([Per07], Proposition 2.13)**.** *The families $(\mathcal{L}_i)_{i\in[1,r]}$ and $(Y_i)_{i\in[1,r]}$ are dual to each other.*

Proof. Let $i, j \in [1, r]$. If $i \neq j$, the sheaf $(\mathcal{L}_i)_{|Y_j}$ is trivial. Therefore, the equality $[\mathcal{L}_i] \cdot [Y_j] = 0$ holds. If $i = j$, the sheaf $(\mathcal{L}_i)_{|Y_i}$ is the generating very ample sheaf on $Y_i \cong \mathbb{P}^1$. That proves $[\mathcal{L}_i] \cdot [Y_i] = 1$. □

As the following proposition shows the families $(\mathcal{L}_i)_{i \in [1,r]}$ and $(Y_i)_{i \in [1,r]}$ are bases of $A^1(\widetilde{X}(\widetilde{w}))$ respectively $A_1(\widetilde{X}(\widetilde{w}))$.

Proposition 4.33 ([Per07], Proposition 2.14)**.** *Let $[C_{s(i)}] = 0$ if $s(i)$ does not exist. Then we have the equality $[Y_i] = [C_i] - [C_{s(i)}]$ for all $i \in [1, r]$.*

Proof. The curve C_i is given as follows

$$\begin{aligned} C_i &= \{(x_1, \ldots, x_r) |\ x_j = x_{p(j)} \text{ for all } j \neq i\} \\ &= \{(x_1, \ldots, x_r) |\ x_j \neq 1 \text{ iff } \beta_i = \beta_j \text{ and } j \geq i. \text{ In that case } x_i = x_j\}. \end{aligned}$$

With this description the equation follows without difficulties. □

Having established these results, we are now able to describe the ample divisors and effective curves.

Theorem 4.34 ([Per07], Corollary 2.15)**.** *The closure of the cone of ample divisors in $A^1(\widetilde{X}(\widetilde{w})) \otimes_{\mathbb{Z}} \mathbb{R}$ is generated by $([\mathcal{L}_i])_{i \in [1,r]}$. The cone of effective curves is generated by $([Y_i])_{i \in [1,r]}$. Moreover, all ample divisors are very ample.*

Proof. Let D be any ample divisor. Then $a_i := [D] \cdot [Y_i]$ is a positive integer and the equality $[D] = \sum_{i=1}^r a_i \mathcal{L}_i$ holds. Let $D = \sum_{i=1}^r b_i \mathcal{L}_i$ be a divisor with $b_i > 0$ for all $i \in [1, r]$. Then D induces the composition of the inclusion $\widetilde{X}(\widetilde{w}) \hookrightarrow \prod_{i=1}^r G/P^{\beta_i}$ and the embedding induced by the very ample sheaf $\bigotimes_{i=1}^r \mathcal{O}_{G/P^{\beta_i}}(b_i)$. Therefore, D is very ample. The claim on effective curves follows by duality. □

We finish this section by computing the divisors $\mathcal{L}_i$ in terms of the basis $([\xi_k])_{k \in [1,r]}$. In order to do this we describe the dual basis to $(\xi_i)_{i \in [1,r]}$. For all $i \in [1, r]$ we define a cycle $[\widehat{C}_i] := [C_i] + \sum_{k=i+1}^n \langle \gamma_i^\vee, \gamma_k \rangle [C_k]$.

Proposition 4.35 ([Per05], Lemma 4.5)**.** *The equality $[\widehat{C}_i] \cdot \xi_j = \delta_{i,j}$ holds for all numbers $i, j \in [1, r]$. That means the curves $([\widehat{C}_i])_{i \in [1,r]}$ form a dual basis to $(\xi_i)_{i \in [1,r]}$.*

Proof. This follows from the following computation using Proposition 4.27 and Lemma

4.30, which is still true after switching the (γ_i) and (β_i).

$$\begin{aligned}
[\widehat{C}_i]\cdot\xi_j &= ([C_i] + \sum_{k=i+1}^{n}\langle\gamma_i^\vee,\gamma_k\rangle[C_k])\cdot\xi_j \\
&= \begin{cases} 0 & \text{, if } i > j \\ 1 & \text{, if } i = j \\ \langle\beta_i^\vee,\beta_j\rangle + \sum_{k=i+1}^{j-1}\langle\gamma_i^\vee,\gamma_k\rangle\langle\beta_k^\vee,\beta_j\rangle + \langle\gamma_i^\vee,\gamma_j\rangle & \text{, if } i < j \end{cases} \\
&= \delta_{i,j}
\end{aligned}$$

□

We have the equality $\mathcal{L}_i = \sum_{k\in[1,r]}(\mathcal{L}_i\cdot[\widehat{C}_k])\xi_k$. To get a description of $\mathcal{L}_i$ in terms of the basis $(\xi_k)_{k\in[1,r]}$ we are left to compute $\mathcal{L}_i\cdot[\widehat{C}_k]$ for all $k\in[1,r]$.

Proposition 4.36 ([Per07], Proposition 2.16)**.** *We have the following equality.*

$$\mathcal{L}_i\cdot[\widehat{C}_k] = \begin{cases} 0 & \text{, if } k > i \\ 1 & \text{, if } k = i \\ 1 + \sum\limits_{j=k+1,\beta_j=\beta_i}^{i}\langle\gamma_k^\vee,\gamma_j\rangle & \text{, if } k < i \text{ and } \beta_k = \beta_i \\ \sum\limits_{j=k+1,\beta_j=\beta_i}^{i}\langle\gamma_k^\vee,\gamma_j\rangle & \text{, if } k < i \text{ and } \beta_k \neq \beta_i \end{cases}$$

Proof. An induction on j using Proposition 4.32 and 4.33 yields the equality

$$\mathcal{L}_i\cdot[C_j] = \begin{cases} 1 & \text{, if } i > j \text{ and } \beta_i = \beta_j \\ 0 & \text{, otherwise} \end{cases}.$$

The statement follows immediately. □

The results of this chapters also enable us to compute the Picard group of a Schubert variety in G/P. We will only deal with the case of a maximal parabolic subgroup P. With some more efforts it is however possible to generalize the following theorem to any parabolic subgroup.

Theorem 4.37. *Let P be a maximal parabolic subgroup of G and $X_P(w)\subseteq G/P$ a non trivial Schubert variety. Then the Picard group $\mathrm{Pic}(X_P(w))$ is isomorphic to $\mathbb{Z}$ and generated by a very ample generator.*

Proof. We proved before that there is a very ample generator $\mathcal{O}(1)$ of $\mathrm{Pic}(G/P)$. The pullback $\pi^*(\mathcal{O}(1)_{|X_P(w)})$ is isomorphic to $\mathcal{L}_r$. Since $A_*(X_P(w))$ (Theorem 3.8) is a free abelian group, the subgroup $\mathrm{Pic}(X_P(w))$ is also free abelian. Therefore, we only need to prove that any invertible sheaf on $X_P(w)$ is a multiple of $\mathcal{O}(1)_{|X_P(w)}$.

The map $\pi^* : \mathrm{Pic}(X_P(w)) \to \mathrm{Pic}(\widetilde{X}(\widetilde{w}))$ is injective. Indeed, by the projection formula and normality of Schubert varieties (Theorem 3.7) the equality $\pi_*\pi^*(\mathcal{L}) = \mathcal{L}$ holds for all $\mathcal{L} \in \mathrm{Pic}(X_P(w))$. Thus, for any $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(X_P(w))$ we get the equivalence

$$\mathcal{L} = \mathcal{L}' \Leftrightarrow \ \pi^*(\mathcal{L}) \cdot [Y_i] = \pi^*(\mathcal{L}') \cdot [Y_i] \ \forall i \in [1, r].$$

Let $\mathcal{L}$ be any line bundle of $X_P(w)$. Then by definition of $[Y_i]$ we have $\pi_*[Y_i] = 0$ for all $i \in [1, r-1]$. The map $\pi_* : A_0(\widetilde{X}(\widetilde{w})) \to A_0(X_P(w))$ is an isomorphism. The projection formula shows $\pi_*(\pi^*\mathcal{L} \cdot [Y_i]) = 0$ for all $i \in [1, r-1]$. Therefore, we get $\mathcal{L} \cong (\mathcal{O}(1)_{|X_P(w)})^{\otimes(\pi^*\mathcal{L}\cdot[Y_r])}$. □

5 Cominuscule Schubert Varieties

5.1 Minuscule Weights

After establishing more general results in the previous parts we are now going to focus on (co)minuscule Schubert varieties. This chapter consists of a recall on (co)minuscule weights. For all root systems the notation of [Bou68] is used.

Definition 5.1. Let ω be a fundamental weight corresponding to a simple root α.

(i) We call ω **minuscule** if $\langle \beta^\vee, \omega \rangle \leq 1$ for all $\beta \in R^+$.

(ii) We call ω **cominuscule** if the fundamental weight $\omega^\vee$ corresponding to $\alpha^\vee \in R^\vee$ is minuscule.

We will classify all minuscule and cominuscule weights by giving an easier description of cominuscule weights.

Proposition 5.2. *(i) Let ω be a fundamental weight corresponding to a simple root α_ω and $\widetilde{\alpha} = \sum_{\alpha \in S} n_\alpha \alpha$ be the highest root of R. Then ω is cominuscule if and only if $n_{\alpha_\omega} = 1$.*

(ii) The following table shows all the minuscule and cominuscule weights.

Type	*Minuscule weights*	*Cominuscule weights*
A_n	$\omega_1, \ldots, \omega_n$	$\omega_1, \ldots, \omega_n$
B_n	ω_n	ω_1
C_n	ω_1	ω_n
D_n	$\omega_1, \omega_{n-1}, \omega_n$	$\omega_1, \omega_{n-1}, \omega_n$
E_6	ω_1, ω_6	ω_1, ω_6
E_7	ω_7	ω_7
E_8	*none*	*none*
F_4	*none*	*none*
G_2	*none*	*none*

Proof. The inequality $n_{\alpha_\omega} \geq 1$ always holds. Furthermore, the definition of ω shows $\langle \omega^\vee, \widetilde{\alpha} \rangle = n_{\alpha_\omega}$. On the one hand, $\omega^\vee$ being minuscule implies $n_{\alpha_\omega} = \langle \omega^\vee, \widetilde{\alpha} \rangle \leq 1$. On the other hand, $n_{\alpha_\omega} = 1$ implies the inequality

$$\langle \omega^\vee, \beta \rangle = m_{\alpha_\omega} \leq n_{\alpha_\omega} = 1$$

for all $\beta = \sum_{\alpha \in S} m_\alpha \alpha \in R^+$. Indeed, we proved that ω is cominuscule if and only if $n_{\widetilde{\alpha}} = 1$.

To get the table of all (co)minuscule weights, just look up the highest roots of all the root systems in [Bou68]. □

The resolutions we will discuss in this thesis have been fully understood in the minuscule case due to [Per07]. We are going to generalize those results to the cominuscule case. The last proposition shows that the number of cases to check is rather limited.

Definition 5.3. Let ω be a minuscule (resp. cominuscule) fundamental weight corresponding to a simple root α.

(i) We define the maximal parabolic subgroup P_ω by $\Sigma(P_\omega) = \{\alpha\}$.

(ii) The homogeneous space G/P_ω is called minuscule (resp. cominuscule) and all its Schubert varieties are also called minuscule (resp. cominuscule).

(iii) An element $w \in W$ is said to be minuscule (resp. cominuscule) with respect to ω if $P_\omega \subseteq P^w$.

Remember the definition $\Sigma(P^w) = \{\alpha \in S \mid l(ws_\alpha) < l(w)\}$. Using Proposition 3.6 we get that the minuscule (resp. cominuscule) elements with respect to ω are exactly the minimal representatives modulo W_{P_ω}. Next we derive a useful lemma which enables us to get some information about W/W_{P_ω}. Note that we can focus on the minuscule case because $W(R) \cong W(R^\vee)$. Later on there will be a summary of the information obtained for the cominuscule case.

Lemma 5.4 ([Per07], Fact 3.8)**.** *Let $\widetilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression and ω be a minuscule fundamental weight. Set $w_i = s_{\beta_i} \cdots s_{\beta_r}$ for $i \in [1, r]$ and $w_{r+1} = 1$. Then the following are equivalent.*

(i) The element w is minuscule with respect to ω.

(ii) For all $i \in [2, r+1]$ we have the equality $\langle \beta_{i-1}^\vee, w_i(-\omega)\rangle = -1$.

(iii) If $i \in [1, r+1]$ we obtain $w_i(-\omega) = -\omega + \beta_r + \ldots + \beta_i$.

Proof. The equivalence of (ii) and (iii) is obvious. Due to Lemma 3.4, w being minuscule is equivalent to $N_{P_\omega}(w_i) < N_{P_\omega}(s_{\beta_{i-1}} w_i)$ for all $i \in [2, r+1]$. That is true if and only if $w_i^{-1}(\beta_{i-1}) \in R^+ \backslash R^+_{P_\omega}$, which is equivalent to $\langle \beta_{i-1}^\vee, w_i(\omega)\rangle = \langle w_i^{-1}(\beta_{i-1})^\vee, \omega\rangle > 0$. We can conclude the proof due to ω being minuscule. □

5.2 Quivers of Minuscule Schubert Varieties

In this chapter we give an explicit description of all minuscule elements $w \in W$ in terms of quivers. The next proposition describes the shape of minuscule quivers.

Lemma 5.5 ([Per07], Proposition 4.1)**.** *Let $\widetilde{w} = (s_{\beta_1}, \dots, s_{\beta_r})$ be a reduced expression and ω a fundamental weight. Then w is minuscule with respect to ω if and only if the following three conditions hold.*

(i) The element β_r is the unique simple root such that $\langle \beta_r^\vee, \omega \rangle = 1$.

(ii) Let $i < r$ be a vertex of the quiver such that $s(i)$ does not exist. Then there is a unique arrow from i to a vertex k and we have $\langle \beta_i^\vee, \beta_k \rangle = -1$.

(iii) Let $i < r$ be a vertex of the quiver such that $s(i)$ exists. Then there are two possibilities. Either there are two vertices k_1, k_2 with an arrow coming from i or there is a unique vertex k with an arrow coming from i. In the first case,

$$\langle \beta_i^\vee, \beta_{k_1} \rangle = \langle \beta_i^\vee, \beta_{k_2} \rangle = -1.$$

In the second case,

$$\langle \beta_i^\vee, \beta_k \rangle = -2.$$

Proof. Let w be cominuscule. We start by showing that all three conditions hold.

(i) By lemma 5.4 the equality $\langle \beta_r^\vee, \omega \rangle = 1$ holds.

(ii) Since $s(i)$ does not exist and $i \neq r$, we get $\beta_i \neq \beta_r$. Using lemma 5.4 we get

$$\begin{aligned} \sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k \rangle &= \langle \beta_i^\vee, -\omega + \sum_{k=i+1}^{r} \beta_k \rangle \\ &= \langle \beta_i^\vee, w_{i+1}(-\omega) \rangle \\ &= -1. \end{aligned}$$

Again using that $s(i)$ does not exists we have $\langle \beta_i^\vee, \beta_k \rangle \in \{0, -1\}$ for all $k \in [i+1, r]$. That concludes part (i).

(iii) In the same way as before we prove the equality

$$\sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k \rangle = \sum_{k=s(i)+1}^{r} \langle \beta_i^\vee, \beta_k \rangle = \begin{cases} -1 & \text{, if } \beta_i \neq \beta_r \\ 0 & \text{, if } \beta_i = \beta_r. \end{cases}$$

Either way we get

$$2 + \sum_{k=i+1}^{s(i)-1} \langle \beta_i^\vee, \beta_k \rangle = \sum_{k=i+1}^{s(i)} \langle \beta_i^\vee, \beta_k \rangle = 0.$$

This concludes part (iii).

To prove the other direction we start by showing the equality

$$\langle \beta_i^\vee, \omega\rangle - \sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k\rangle = 1 \tag{5.1}$$

for all $i \in [1, r]$ by descending induction on i. For $i = r$ the equality $\langle \beta_i^\vee, \omega\rangle = 1$ holds due to (i). If $s(i)$ doesn't exist and $i \neq r$, condition (ii) implies (5.1). If $s(i)$ exists, condition (iii) implies

$$\sum_{k=i+1}^{s(i)} \langle \beta_i^\vee, \beta_k\rangle = 0.$$

We can conclude because by induction the equality (5.1) holds for $s(i)$.

To proof that w is minuscule we show part (iii) of Lemma 5.4 also by descending induction on i. The equality $w_{r+1}(-\omega) = -\omega$ always hold. For $i \leq r$ the equality (5.1) and the induction hypothesis yield

$$\begin{aligned} w_i(-\omega) &= s_{\beta_i} w_{i+1}(-\omega) \\ &= s_{\beta_i}(-\omega + \beta_r + \ldots + \beta_{i+1}) \\ &= -\omega + \beta_r + \ldots + \beta_{i+1} + (\langle \beta_i^\vee, \omega\rangle - \sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k\rangle)\beta_i \\ &= -\omega + \beta_r + \ldots + \beta_i. \end{aligned}$$

□

With this description of minuscule elements we are able to proof several useful facts.

Theorem 5.6. *Let ω be a minuscule (resp. cominuscule) fundamental weight.*

(i) Minuscule (resp. cominuscule) elements have a unique reduced expression modulo commutation relations.

(ii) The Bruhat order in W/W_{P_ω} is the same as the weak Bruhat order.

Proof. (i) The only equations in W which could change a reduced expression beyond commutation are given by the braid relations $s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha$ and $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ for simple roots α and β. The relation $s_\alpha s_\beta s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta s_\alpha s_\beta$ does not occur because there is no (co)minuscule weight in the root system of type G_2. We may assume that the length of the root α is smaller or equal to the root β. Therefore, the equality $\langle \beta^\vee, \alpha\rangle = -1$ holds. If any of the two relations is applicable, there is a reduced expression where $s_\alpha s_\beta s_\alpha$ occurs. But Lemma 5.5 shows that there has to be a second simple reflection in between the two s_α.
(ii) The Bruhat order in the Weyl group W is generated by the relations $u < v$, where $l(v) = l(u)+1$ and u is obtained from $v \in W$ by removing a simple reflection in a reduced

expression (see [Bjo05] Theorem I.2.2.6). The weak Bruhat order in W is generated by the relations $u < v$, where u is obtained from $v \in W$ by removing a simple reflection from the left in any reduced expression. Let $\widetilde{w} = (s_{\beta_1}, \dots, s_{\beta_r})$ be a reduced expression of a minimal representative $w \in W$ modulo W_{P_ω}. It suffices to prove that removing any simple reflection from $\widetilde{w}$ which is not at the left (modulo commutation) does not yield a reduced expression. Let s_{β_j} be a simple reflection which is not at the left for $j \in [1, r]$. Then there is an $i \in [1, r]$ with an arrow to j. By removing j Lemma 5.5 (ii) or (iii) fails for i if $j \neq r$. If however $j = r$, then removing j yields a quiver not satisfying part (i) of the lemma. □

Lemma 5.7. *Every expression obtained from a quiver satisfying the conditions of Lemma 5.5 is reduced.*

Proof. If the expression would not be reduced, we could apply either a braid relation or a relation $s_\alpha^2 = 1$ for some $\alpha \in S$. As in the previous proof, that does not occur by Lemma 5.5. □

It is desirable to link this description of minuscule elements to the Bruhat order.

Definition 5.8. (i) We define a partial order on the vertices of the quiver generated by the relations $j \prec i$ if there is an arrow from i to j.

(ii) A subset I of a partially ordered set $(P, \leq)$ is called an **order ideal** if the following condition holds
$$(x \in I,\ x \geq y \in P) \Rightarrow y \in Y.$$

This partial order is not the one defined in [Per07], but the reversed one. Note, that the order ideals for our partial order are subquivers and have an induced poset structure.

Theorem 5.9 ([Per07], Proposition 4.5). *Let ω be a minuscule fundamental weight and Q_ω the quiver of the minimal representative w_ω of the maximal element in W/W_{P_ω}. The quotient W/W_{P_ω} and the order ideals of Q_ω are isomorphic as partially ordered sets.*

Proof. By taking the corresponding expression as described before in Section 3.3 we define a map $\varphi : \{\text{order ideals of } Q_\omega\} \to W$. Furthermore, we have the projection map $\pi : W \to W/W_{P_\omega}$. We are going to prove that $\pi \circ \varphi$ is an isomorphism of partially ordered sets.

We already know that there is at most one quiver to each reduced expression. Every order ideal clearly satisfies the conditions of Lemma 5.5. Therefore, ϕ maps onto the set of minuscule elements with respect to ω. By Theorem 5.6 (i) injectivity follows.

Next we prove the surjectivity of $\pi \circ \varphi$. Pick any element $w \in W$ which is minuscule with respect to ω. Then $w \leq w_\omega$ and Theorem 5.6 (ii) implies the existence of

$\beta_1, \ldots, \beta_s, \ldots, \beta_r \in S$ such that $w_\omega = s_{\beta_1} \cdots s_{\beta_s} w$ and $w = s_{\beta_{s+1}} \ldots s_{\beta_r}$. This implies that the quiver corresponding to w is an order ideal of Q_ω proving surjectivity.

The assertion that we have an isomorphism of partially ordered sets is now a simple consequence of Theorem 5.6 (ii). □

Finally, we are going to fix some further notation for some (co)minuscule element $w \in W$ with respect to a (co)minuscule fundamental weight ω. Recall the equality

$$\Sigma(P_w) = \{\alpha \in S | \ \overline{s_\alpha w} > \overline{w} \text{ for the Bruhat order in } W/W_{P^w}\}.$$

Definition 5.10. (i) A maximal vertex of Q_w is called a **peak**. We denote the set of peaks of Q_w by $p(Q_w)$.

(ii) We call a vertex $i \in Q_\omega$ a **hole** of Q_w if one of the following two conditions is satisfied.

a) The vertex i is in Q_w, $p(i) \notin Q_w$ and $\beta_i \in \Sigma(P_w)$.

b) The vertex i is not in Q_w, $s(i)$ does not exist and $\beta_i \in \Sigma(P_w)$.

The holes satisfying the second condition are called **virtual holes**. We denote the set of holes of Q_w by $\text{Holes}(Q_w)$.

(iii) The **height** $h(i)$ of a vertex i of Q_w is the largest integer n such that there is a path consisting of $n-1$ arrows from i to the unique smallest vertex in Q_w

By the definition it is clear that $\beta(\text{Holes}(Q_w)) = \Sigma(P_w)$. We give a purely combinatorial description of a hole in the minuscule case. The cominuscule case is dealt with in the next section.

Proposition 5.11. *Let ω be a fundamental weight and $w \neq 1$. A vertex i of Q_ω is a hole of Q_w if and only if one of the following two conditions is satisfied.*

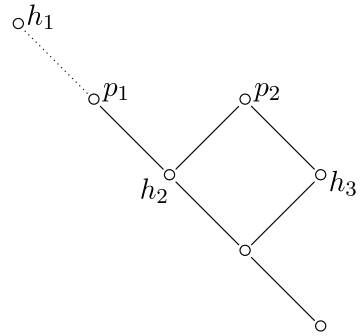
(i) The vertex i is in Q_w, $p(i) \notin Q_w$ and there are between one or two vertices $k_1, k_2 \succ i$ such that $\langle \beta_i^\vee, \beta_{k_j} \rangle \neq 0$ for $j = 1, 2$. Moreover, we need to have the following equality for $j \in \{1, 2\}$

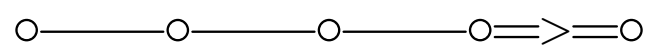
$$\langle \beta_i^\vee, \beta_{k_j} \rangle = \begin{cases} -1 & , \text{ if } k_1 \neq k_2 \\ -2 & , \text{ if } k_1 = k_2. \end{cases}$$

(ii) The vertex i is not in Q_w, $s(i)$ does not exist and $\beta_i \in \partial \operatorname{Supp}(w)$.

Proof. The two conditions are exactly the ones that are needed such that part (ii) or (iii) of Lemma 5.5 is still fulfilled after adding the vertex $p(i)$ or i to the quiver Q_ω. □

Example 5.12. Let us go back to our previous Example 3.15. Instead of C_5 we will work over B_5. This doesn't change anything on the quiver. It is only necessary, because we have not really dealt with the cominuscule case. The holes are h_1, h_2, h_3 and the peaks are p_1, p_2.





The vertex h_1 is the only virtual hole. The heights of p_1, p_2, h_1, h_2, h_3 are 4, 4, 5, 3, 3.

Lemma 5.13. *Let $X_P(w')$ be a Schubert subvariety of a (co)minuscule Schubert variety $X_P(w)$ stable under P_w. Then*

$$\beta(\text{Holes}(Q_{w'})) \subseteq \beta(\text{Holes}(Q_w)).$$

Proof. We have the inclusion $P_w \subseteq P_{w'}$. Therefore, $\Sigma(P_{w'}) \subseteq \Sigma(P_w)$ holds. That implies the lemma. □

5.3 Quivers of Cominuscule Schubert Varieties

After having established many results on minuscule elements of the Weyl group we are going to rewrite them for the cominuscule case. All statements in this section are obvious reformulations. All proofs follow from the fact that $W(R) \cong W(R^\vee)$ and are omitted.

Lemma 5.14. *Let $\widetilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression and ω a fundamental weight. Then w is cominuscule with respect to ω if and only if the following three conditions hold.*

(i) *The element β_r is the unique simple root such that $\langle \beta_r^\vee, \omega \rangle = 1$.*

(ii) *Let $i < r$ be a vertex of the quiver such that $s(i)$ does not exist. Then there is a unique arrow from i to a vertex k and we have $\langle \beta_k^\vee, \beta_i \rangle = -1$.*

(iii) *Let $i < r$ be a vertex of the quiver such that $s(i)$ exists. Then there are two possibilities. Either there are two vertices k_1, k_2 with an arrow coming from i or there is a unique vertex k with an arrow coming from i. In the first case,*

$$\langle \beta_{k_1}^\vee, \beta_i \rangle = \langle \beta_{k_2}^\vee, \beta_i \rangle = -1.$$

In the second case,

$$\langle \beta_k^\vee, \beta_i \rangle = -2.$$

Theorem 5.15. *Let ω be a cominuscule fundamental weight.*

(i) Cominuscule elements have a unique reduced expression modulo commutation relations.

(ii) The Bruhat order in W/W_{P_ω} is the same as the weak Bruhat order.

Lemma 5.16. *Every expression obtained from a quiver satisfying the conditions of Lemma 5.14 is reduced.*

Theorem 5.17. *Let ω be a cominuscule fundamental weight and Q_ω the quiver of the minimal representative w_ω of the maximal element in W/W_{P_ω}. The Weyl group W/W_{P_ω} and the order ideals of Q_ω are isomorphic as partially ordered sets.*

The definitions of a peak, a hole and the height remain the same in the cominuscule case. However, the combinatoric description of a hole changes.

Proposition 5.18. *Let ω be a cominuscule fundamental weight and Q_ω the quiver of the minimal representative w_ω of the maximal element in W/W_{P_ω}. Furthermore, $w \in W\backslash\{1\}$ is cominuscule with respect to ω. Then a vertex i of Q_ω is a hole of Q_w if and only if one of the following two conditions is satisfied.*

(i) The vertex i is in Q_w, $p(i) \notin Q_w$ and there are between one or two vertices $k_1, k_2 \succ i$ such that $\langle \beta_i^\vee, \beta_{k_j} \rangle \neq 0$ for $j = 1, 2$. Moreover, we need to have the following equality for $j \in \{1, 2\}$

$$\langle \beta_{k_j}^\vee, \beta_i \rangle = \begin{cases} -1 & , \text{ if } k_1 \neq k_2 \\ -2 & , \text{ if } k_1 = k_2 \end{cases}$$

(ii) The vertex i is not in Q_w, $s(i)$ does not exist and $\beta_i \in \partial\,\mathrm{Supp}(w)$.

Lemma 5.19. *Let $X_P(w')$ be a Schubert subvariety of a cominuscule Schubert variety $X_P(w)$ stable under P_w. Then*

$$\beta(\mathrm{Holes}(Q_{w'})) \subseteq \beta(\mathrm{Holes}(Q_w)).$$

6 Divisors of Cominuscule Schubert Varieties

For the rest of this thesis we are only going to deal with cominuscule Schubert varieties. There are two cases which are not minuscule. The first one is C_n with weight ω_n, while the second one is B_n with weight ω_1. Furthermore, we also need some results in the A_n case. All the following results have corresponding statements in the minuscule case. They are to be found in [Per07]. In this section we describe further properties of divisors. At first, we need a description of the quivers of the maximal elements. They are given by the maximal quivers satisfying the conditions of Lemma 5.14.

Proposition 6.1. *(i) If G has root system C_n and P is the maximal parabolic corresponding to the fundamental weight ω_n, the quiver of the maximal element in W/W_P can be described as follows. The vertices are given by the set*

$$Q := \{(x,y) \in [1,n]\times[1,2n-1] \mid x+y \equiv n+1 \pmod 2,\ x+y \geq n+1,\ y-x \leq n-1\}.$$

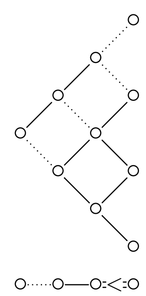
The coloring $\beta : Q \to S$ is given by $(x,y) \mapsto \alpha_x$. We define two subsets of Q by

$$A := \{(x,y) \in Q \mid x+y = n+1 \text{ and } (x,y) \neq (n,1)\},$$
$$B := \{(x,y) \in Q \mid x = n \text{ and } (x,y) \neq (n,1)\}.$$

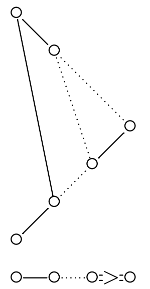
Then we define the following arrows.

- *If $(x,y) \in A$, there is an arrow $(x,y) \to (x+1, y-1)$.*
- *If $(x,y) \in B$, there is an arrow $(x,y) \to (x-1, y-1)$.*
- *If $(x,y) \in Q\backslash(A \cup B \cup \{(n,1)\})$, there are two arrows $(x,y) \to (x-1,y-1)$ and $(x,y) \to (x+1,y-1)$.*

This quiver looks as follows.



(ii) If G has root system B_n and P is the maximal parabolic corresponding to the fundamental weight ω_1, then $w = s_{\alpha_1} \cdots s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$ is the maximal element in W/W_P. The quiver Q_w looks as follows.



(iii) If G has root system A_n and P is the maximal parabolic corresponding to the fundamental weight ω_k for $k \in [1, n]$, the quiver of the maximal element in W/W_P can be described as follows. The vertices are given by the set

$$Q := \{(x, y) \in [1, n]^2 \mid x+y \equiv k+1 \pmod 2,\ x+y \in [k+1, 2n-k+1],\ y-x \in [1-k, k-1]\}.$$

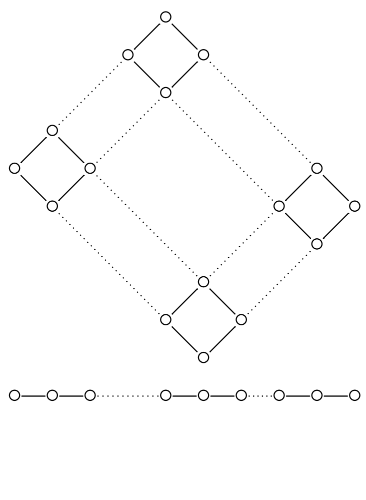
The coloring $\beta : Q \to S$ is given by $(x, y) \mapsto \alpha_x$. We define two subset of Q by

$$A := \{(x, y) \in Q \mid x + y = k + 1 \text{ and } (x, y) \neq (k, 1)\},$$
$$B := \{(x, y) \in Q \mid y - x = 1 - k \text{ and } (x, y) \neq (k, 1)\}.$$

Then we define the following arrows.

- *If $(x, y) \in A$, there is an arrow $(x, y) \to (x + 1, y - 1)$.*
- *If $(x, y) \in B$, there is an arrow $(x, y) \to (x - 1, y - 1)$.*
- *If $(x, y) \in Q \backslash (A \cup B \cup \{(k, 1)\}$, there are two arrows $(x, y) \to (x - 1, y - 1)$ and $(x, y) \to (x + 1, y - 1)$.*

This quiver looks as follows.



6.1 The C_n-Case

In this chapter we will do some computations in the C_n case. In Section 4.2 we computed properties of divisors in terms of the roots $(\gamma_i)_{i\in[1,r]}$. Having concrete quivers we are now able to compute this even more explicitly. Let $w = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression of a cominuscule element with respect to ω_n.

Proposition 6.2. *Let Q be as in Proposition 6.1, $Q_w \subseteq Q$ the quiver corresponding to w and $i = (v, z) \in Q_w$. We define the set*

$$C := \{j = (x, y) \in Q_w \mid j \succeq i \text{ and } \langle \gamma_j^\vee, \gamma_i \rangle \neq 0\}.$$

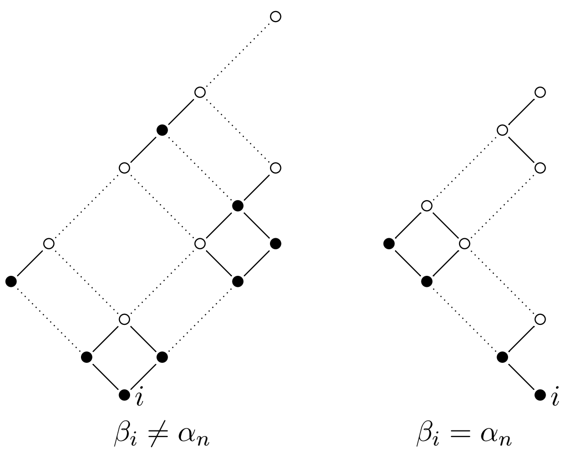
Then we have the equality

$$C = \{j = (x, y) \in Q_w \mid j \succeq i \text{ and } (y - x = z - v \text{ or } x + y = v + z \text{ or } x + y = z - v + 2n)\}.$$

Moreover, for all $j \in C$ we have the equality

$$\langle \gamma_j^\vee, \gamma_i \rangle = \begin{cases} 2 & , \text{ if } \gamma_j \text{ is shorter than } \gamma_i \\ 1 & , \text{ otherwise.} \end{cases}$$

Remark 6.3. Assuming $Q_w = Q$ the following diagram shows the subquiver of Q consisting of all vertices $j \in Q$ with $j \succeq i$. The filled vertices and all the vertices on dotted lines between them are exactly the ones in C.



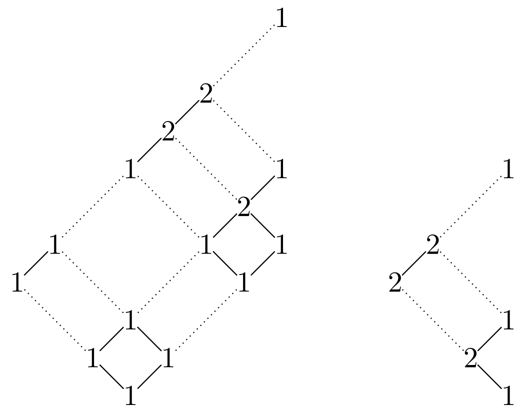
Proof. We have the equality $\langle \gamma_j^\vee, \gamma_i \rangle = \langle \beta_j^\vee, s_{\beta_j} \ldots s_{\beta_{i-1}}(\beta_i) \rangle$. Therefore, it suffices to do the calculation for the maximal element in W/W_P. Let $f : \{j = (x, y) \in Q \mid j \succeq i\} \to \mathbb{Z}$ be the map defined by

$$j \mapsto \text{coefficient of } \beta_j \text{ in } s_{\beta_j} \cdots s_{\beta_{i-1}}(\beta_i).$$

An induction on the ordering of Q yields

$$f(x,y) = \begin{cases} 1 & \text{, if } x = n \\ 1 & \text{, if } x+y \in [v+z, z-v+2n-2] \text{ and } y-x \in [z-v, n-1] \\ 2 & \text{, otherwise.} \end{cases}$$

The following diagram shows the values of f written on the place of the vertices of the quiver.



$\beta_i \neq \alpha_n$ $\qquad\qquad$ $\beta_i = \alpha_n$

From this it is easy to calculate $\langle \gamma_j^\vee, \gamma_i \rangle$ and the proposition follows. □

The proposition was formulated for a fixed i. We can also formulate it for a fixed j as follows. Let $Q_w \subseteq Q$ be the quiver corresponding to w and let $j = (x,y) \in Q_w$. We define the set

$$D := \{i = (v,z) \in Q_w \mid i \preceq j \text{ and } \langle \gamma_j^\vee, \gamma_i \rangle \neq 0\}.$$

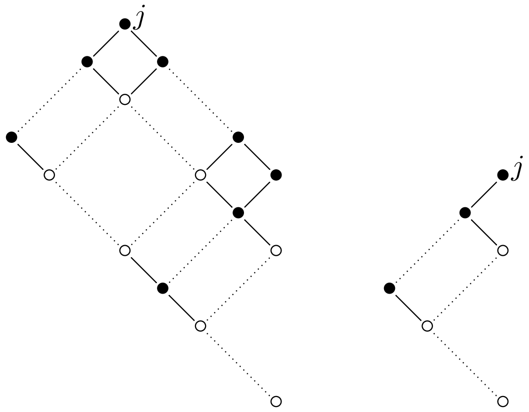
Then we have the equality

$$D = \{i = (v,z) \in Q_w \mid i \preceq j \text{ and } (z-v = y-x \text{ or } v+z = x+y \text{ or } z-v = x+y-2n)\}.$$

Moreover, for all $j \in D$ we have the equality

$$\langle \gamma_j^\vee, \gamma_i \rangle = \begin{cases} 2 & \text{, if } \gamma_j \text{ is shorter than } \gamma_i \\ 1 & \text{, otherwise.} \end{cases}$$

The following diagram shows the subquiver of Q_w consisting of all vertices $i \in Q_w$ with $i \preceq j$. The filled vertices and all the vertices on dotted lines between them are exactly the ones in D.



$\beta_j \neq \alpha_n$ $\qquad$ $\beta_j = \alpha_n$

Together with Proposition 4.36 we can calculate $\mathcal{L}_r$.

Corollary 6.4. *We have the equality*

$$\mathcal{L}_r = \sum_{i \in Q_w, \beta_i \neq \alpha_n} 2\xi_i + \sum_{i \in Q_w, \beta_i = \alpha_n} \xi_i.$$

As another consequence of Proposition 6.2 we can calculate the canonical divisor of $\widetilde{X}(\widetilde{w})$. Recall the definition of the height in Section 5.2. For notational purposes we define the **adjusted height** of each vertex i by

$$h'(i) := \begin{cases} h(i) & \text{, if } \beta_i \neq \alpha_n \\ h(i) + 1 & \text{, if } \beta_i = \alpha_n. \end{cases}$$

Corollary 6.5. *We have the equality*

$$-K_{\widetilde{X}(\widetilde{w})} = \sum_{i \in Q_w, \beta_i \neq \alpha_n} (h'(i) + 2)\xi_i + \sum_{i \in Q_w, \beta_i = \alpha_n} \frac{h'(i) + 2}{2} \xi_i.$$

Proof. This can be computed using the description $-K_{\widetilde{X}(\widetilde{w})} = \sum_{k=1}^{r} \left(\sum_{i=k}^{r} \langle \gamma_k^\vee, \gamma_i \rangle \right) \xi_k$. □

The Picard group and the divisor class group of $X_P(w)$ can be described in terms of the elements $D_i = \pi_* \xi_i$ for $i \in p(Q_w)$.

Proposition 6.6. *The divisor class group of $X_P(w)$ is the free abelian group generated by the classes D_i for $i \in p(Q_w)$. The Picard group is free and generated by the class*

$$\mathcal{L}(w) := \pi_*(\mathcal{L}_r) = \sum_{\substack{i \in p(Q_w) \\ \beta_i \neq \alpha_n}} 2D_i + \sum_{\substack{i \in p(Q_w) \\ \beta_i = \alpha_n}} D_i.$$

Proof. The claim about the divisor class group follows from Theorem 3.8. However, for divisors there is the following easier argument, than what was done in [FMSS95].

The complement of the open Schubert cell BwP/P is given by the union of the divisorial Schubert divisors. There is an $n \in \mathbb{N}$ such that BwP/P is isomorphic to the affine space $\mathbb{A}^n$. Because the divisor class group of $\mathbb{A}^n$ is trivial, the group $A^1(X_P(w))$ is generated by the classes of the divisorial Schubert varieties. Since we are in the cominuscule case, they are given by $[X_P(s_{\beta_i}w)] = \pi_*(\xi_i)$ for $i \in p(Q_w)$. Let $\sum_{i \in p(Q_w)} a_i D_i = 0$ be some relation in the divisor class group. Then there is a rational function f on $X(w)$ with no zero or pole on $BwP/P \cong \mathbb{A}^n$. Therefore, f is constant which implies $a_i = 0$ for all $i \in p(Q_w)$.

We proved in Theorem 4.37 that the Picard group of $X_P(w)$ is isomorphic to $\mathbb{Z}$ and is generated by the element $\pi_*(\mathcal{L}_r) = \mathcal{O}_{G/P_r}(1)_{|X_P(w)}$. The dimension of $\pi(Z_i)$ is different from the dimension of Z_i if $i \notin p(Q_w)$. Therefore, $\pi_*(\xi_i) = 0$ for all $i \notin p(Q_w)$. That concludes the proof. □

As a consequence we can determine whether $X_P(w)$ is locally $\mathbb{Q}$-factorial.

Corollary 6.7. *The Schubert variety $X_P(w)$ is locally $\mathbb{Q}$-factorial if and only if the quiver Q_w has a unique peak. Furthermore, it is locally factorial if and only if Q_w has a unique peak i such that $\beta_i = \alpha_n$.*

For any normal variety the canonical sheaf can be defined as the extension of the canonical sheaf of its smooth locus. See for example [BK05] Remark 1.3.12 for more details. Therefore, we can speak about the canonical divisor $K_{X_P(w)}$. To actually compute it we need to recall the following facts.

Fact 6.8 ([BK05], Lemma 3.4.2)**.** *Let $f : X \to Y$ be a rational resolution of a variety Y. Then Y is Cohen-Macaulay with canonical divisor $f_*(K_X)$.*

Fact 6.9 ([BK05], Theorem 3.4.3)**.** *The Bott-Samelson resolution is rational.*

In particular, the equality $K_{X_P(w)} = \pi_*(K_{\widetilde{X}(\widetilde{w})})$ holds. We define $h'(w) = \max_{i \in Q_w} h'(i)$. As a consequence of Corollary 6.5 and Proposition 6.6, the following proposition follows immediately.

Proposition 6.10. *We have the equalities*

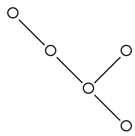
$$\begin{aligned} -K_{X_P(w)} &= \sum_{\substack{i \in p(Q_w) \\ \beta_i \neq \alpha_n}} (h'(i)+2)D_i + \sum_{\substack{i \in p(Q_w) \\ \beta_i = \alpha_n}} \frac{h'(i)+2}{2} D_i \\ &= \frac{h'(w)+2}{2}\mathcal{L}(w) + \sum_{\substack{i \in p(Q_w) \\ \beta_i \neq \alpha_n}} (h'(i)-h'(w))D_i + \sum_{\substack{i \in p(Q_w) \\ \beta_i = \alpha_n}} \frac{h'(i)-h'(w)}{2} D_i. \end{aligned}$$

This enables us to determine whether $X_P(w)$ is $\mathbb{Q}$-Gorenstein.

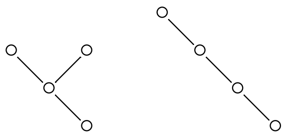
Corollary 6.11. *The Schubert variety $X_P(w)$ is $\mathbb{Q}$-Gorenstein if and only if $h'(i) = h'(j)$ for all peaks i and j of Q_w. Moreover, it is Gorenstein if and only if $h'(i) = h'(j)$ for all peaks i, j of Q_w and $h'(w)$ is even.*

Let us give some examples.

Example 6.12. Let G have root system C_4 and $w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}$. The quiver Q_w is given as follows.



The next diagram shows the quivers of the two Schubert divisors.



We have the equalities $\mathcal{L}(w) = 2D_1 + D_3$ and $K_{X_P(w)} = 6D_1 + 3D_3$. Therefore, $X_P(w)$ is Gorenstein, but not locally $\mathbb{Q}$-factorial. The example $w' = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}$ yields a Schubert variety, that is locally $\mathbb{Q}$-factorial and $\mathbb{Q}$-Gorenstein, but is neither locally factorial nor Gorenstein.

6.2 The B_n-Case

We will now prove corresponding results in the B_n case. Let $w = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression of a minuscule element with respect to ω_1.

Proposition 6.13. *Let $j \in Q_w$. We define the set*

$$A := \{i \in Q_w \mid i \preceq j \text{ and } \langle \gamma_j^\vee, \gamma_i \rangle \neq 0\}.$$

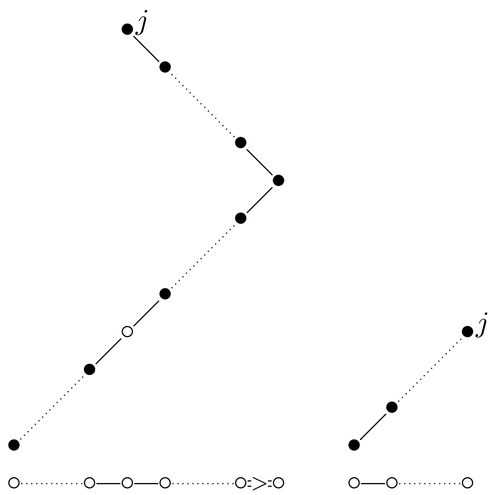
Then the equality

$$A = \{i \in Q_w \mid i \preceq j \text{ and } (\beta_i \neq \beta_j \text{ or } i = j)\}$$

holds. Moreover, for all $i \in A$ we have the equality

$$\langle \gamma_j^\vee, \gamma_i \rangle = \begin{cases} 2 & , \text{ if } \gamma_j \text{ is shorter than } \gamma_i \\ 1 & , \text{ otherwise.} \end{cases}$$

Remark 6.14. The following two diagrams show the subquiver of Q_w consisting of all vertices $i \in Q_w$ with $i \preceq j$. The first one deals with the case, where an $s \in Q_w$ exists with $\beta_s = \alpha_n$ and $j \succ s$. The second one deals with the remaining case. The filled vertices and all the vertices on dotted lines between them are exactly the ones in A.



Proof. The proof works in the same way as in Proposition 6.2. We leave the details to the reader. □

Corollary 6.15. *We have the equality*

$$\mathcal{L}_r = \sum_{i \in Q_w, \beta_i \neq \alpha_n} \xi_i + \sum_{i \in Q_w, \beta_i = \alpha_n} 2\xi_i.$$

Corollary 6.16. *We have the equality*

$$-K_{\widetilde{X}(\widetilde{w})} = \begin{cases} 2h(s)\xi_s + \sum_{i \succ s} h(i)\xi_i + \sum_{i \prec s}(h(i)+1)\xi_i & \text{, if } \exists s \in Q_w \text{ with } \beta_s = \alpha_n \\ \sum_{i \in Q_w}(h(i)+1)\xi_i & \text{, otherwise.} \end{cases}$$

Proof. This can be computed by using the description $-K_{\widetilde{X}(\widetilde{w})} = \sum_{k=1}^{r} \left(\sum_{i=k}^{r} \langle \gamma_k^\vee, \gamma_i \rangle\right) \xi_k$. □

As before we define $D_i = \pi_* \xi_i$ for $i \in p(Q_w)$. We can compute the Picard group and the divisor class group of $X_P(w)$ in the same way as in Proposition 6.6.

Proposition 6.17. *The divisor class group of $X_P(w)$ is isomorphic to $D_p \cdot \mathbb{Z}$, where p is the unique peak of Q_w. For the Picard group we get*

$$\operatorname{Pic}(X_P(w)) \cong \begin{cases} D_p \cdot \mathbb{Z} & \text{, if } \beta_p \neq \alpha_n \\ 2D_p \cdot \mathbb{Z} & \text{, if } \beta_p = \alpha_n \end{cases}.$$

Corollary 6.18. *The Schubert variety $X_P(w)$ is locally $\mathbb{Q}$-factorial. It is locally factorial if and only if $\beta_p \neq \alpha_n$.*

As in the C_n case the equality $K_{X_P(w)} = \pi_*(K_{\widetilde{X}(\widetilde{w})})$ holds.

Proposition 6.19. *Let p be the unique peak of Q_w. We have the equality*

$$-K_{X_P(w)} = \begin{cases} h(p)D_p & , \text{if } \exists s \in Q_w \text{ with } \beta_s = \alpha_n \text{ and } p \succ s \\ 2h(p)D_p & , \text{if } \beta_p = \alpha_n \\ (h(p)+1)D_p & , \text{otherwise.} \end{cases}$$

Corollary 6.20. *The Schubert variety $X_P(w)$ is Gorenstein.*

6.3 The A_n-Case

Finally, we are going to deal with the A_n case. The proofs work exactly the same way as before. Therefore, we will omit them. Most statements can also be found in more generality in [Per07] for all minuscule elements. Let $w = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression of a minuscule element with respect to ω_k for some $k \in [1, n]$.

Proposition 6.21. *Let $Q_w \subseteq Q$ be the quiver corresponding to w and let $j = (x, y) \in Q_w$. We define the set*

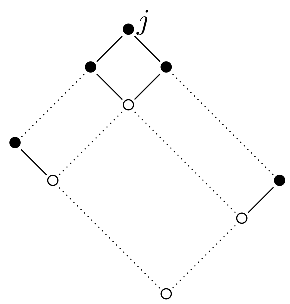
$$C := \{i = (v, z) \in Q_w \mid i \preceq j \text{ and } \langle \gamma_j^\vee, \gamma_i \rangle \neq 0\}.$$

Then we have the equality

$$C = \{i = (v, z) \preceq j \mid z - v = y - x \text{ or } v + z = x + y\}.$$

Moreover, for all $j \in C$ we have the equality $\langle \gamma_j^\vee, \gamma_i \rangle = 1$.

Remark 6.22. The following diagram shows the subquiver of Q_w consisting of all vertices $i \in Q_w$ with $i \preceq j$. The filled vertices and all the vertices on dotted lines between them are exactly the ones in C.



Corollary 6.23. *We have the equality* $\mathcal{L}_r = \sum_{i\in Q_w} \xi_i$.

Corollary 6.24. *We have the equality* $-K_{\widetilde{X}(\widetilde{w})} = \sum_{i\in Q_w}(h(i)+1)\xi_i$.

Again, we define $D_i = \pi_*\xi_i$ for $i \in p(Q_w)$.

Proposition 6.25. *The divisor class group of* $X_P(w)$ *is the free abelian group generated by the classes* D_i *for* $i \in p(Q_w)$. *The Picard group is free and generated by the class*

$$\mathcal{L}(w) := \pi_*(\mathcal{L}_r) = \sum_{i\in p(Q_w)} D_i.$$

Corollary 6.26. *The Schubert variety* $X_P(w)$ *is locally* $\mathbb{Q}$*-factorial if and only if it is locally factorial, which happens if and only if the quiver* Q_w *has a unique peak.*

We define $h(w) = \max_{i\in Q_w} h(i)$.

Proposition 6.27. *We have the equalities*

$$\begin{aligned} -K_{X_P(w)} &= \sum_{i\in p(Q_w)} (h(i)+1)D_i \\ &= (h(w)+1)\mathcal{L}_r + \sum_{i\in p(Q_w)} (h(i)-h(w))D_i. \end{aligned}$$

Corollary 6.28. *The Schubert variety* $X_P(w)$ *is* $\mathbb{Q}$*-Gorenstein if and only if it is Gorenstein, which happens if and only if all peaks of* Q_w *are of the same height.*

7 Resolutions

7.1 Generalization of Bott-Samelson's Construction

In this section we recall a generalization of the Bott-Samelson variety. We follow the lines of [Per07]. For any minuscule $w \in W$ we define projective varieties $\widehat{X}(\widehat{w})$ with a birational morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$. In important special cases this variety has at most terminal singularities. Moreover, the Bott-Samelson resolution will factor through $\widehat{X}(\widehat{w})$. The former one is constructed as a tower of locally trivial $\mathbb{P}^1$-fibrations, while $\widehat{X}(\widehat{w})$ is constructed as a tower of locally trivial fibrations with fibers isomorphic to Schubert varieties. Notice, that the general construction won't need w to be minuscule.

At first we will describe how one step in constructing $\widehat{X}(\widehat{w})$ is done. Let $u \in W$ and Y be a variety on which a parabolic subgroup P_Y containing B acts. We also assume that the inclusion $P^u \cap G_u \subseteq P_Y$ holds. Remember that we have the isomorphism $X_{P^u}(u) \cong \overline{(P_u \cap G_u)u(P^u \cap G_u)}/(P^u \cap G_u)$. That will enable us later to regard $X_{P^u}(u)$ as minuscule, though u is not minuscule in W. We define

$$\widehat{Y}(u) := \overline{(P_u \cap G_u)u(P^u \cap G_u)} \times^{(P^u \cap G_u)} Y.$$

Let $f : \widehat{Y}(u) \to \overline{(P_u \cap G_u)u(P^u \cap G_u)}/(P^u \cap G_u)$ be the map induced by the first projection. The subgroup $P_{\widehat{Y}(u)}$ is defined by

$$\Sigma(P_{\widehat{Y}(u)}) = (\Sigma(P_Y) \cap \operatorname{Supp}(u)^c) \cup \partial \operatorname{Supp}(u) \cup (\Sigma(P_u) \cap \operatorname{Supp}(u)).$$

Lemma 7.1 ([Per07], Lemma 5.1). *(i) The morphism f is a locally trivial fibration with fiber Y.*

(ii) We have the inclusion $P_u \cap G_u \subset P_{\widehat{Y}(u)}$. The action of $P_u \cap G_u$ on $\widehat{Y}(u)$ extends to $P_{\widehat{Y}(u)}$.

Proof. We have the inclusion $\overline{(P_u \cap G_u)u(P^u \cap G_u)} \times^{P_u \cap G_u} Y \hookrightarrow G_u \times^{P_u \cap G_u} Y$. Therefore, it suffices to prove that the latter space is locally trivial over $G_u/(P_u \cap G_u)$. Hence, we may assume u to be the unique maximal element in W/W_{P^u} for the Bruhat order. The morphism f is trivial over the open set BuP^u/P^u and the action of G gives an open covering on which f is trivial. That proves (i).

The first assertion of (ii) follows from $\Sigma_{G_u}(P_u \cap G_u) = \Sigma_{G_u}(P_{\widehat{Y}(u)} \cap G_u)$. We are done with the second statement if we can find a parabolic subgroup $Q \subseteq G$ such that $Q \subseteq P_Y$

and the natural map $\overline{(P_u \cap G_u)u(P^u \cap G_u)}/(P^u \cap G_u) \to \overline{P_{\widehat{Y}(u)}uQ}/Q$ is an isomorphism. We define Q by

$$\Sigma(Q) = (\Sigma(P_Y) \cap \mathrm{Supp}(u)^c) \cup (\Sigma(P^u) \cap \mathrm{Supp}(u)).$$

To get the isomorphism we use Proposition 3.12. In order to do that, we need to use the following inclusions and equalities.

$$\begin{aligned} P_u \cap G_u &= P_{\widehat{Y}(u)} \cap G_u \\ P_{\widehat{Y}(u)} &\subseteq P_u \\ Q \cap G_u &= P^u \cap G_u \end{aligned}$$

□

Definition 7.2. (i) An s-tuple $\widehat{w} = (w_1, \ldots, w_s)$ is called a **generalized decomposition** of w if $w = w_1 \cdots w_n$ holds. Furthermore, it is called **reduced** if the equation $l(w) = \sum_{i=1}^{s} l(w_i)$ is true.

(ii) We define a sequence of parabolic subgroups by $P_s = P_{w_s}$ and

$$\Sigma(P_i) = (\Sigma(P_{i+1}) \cap \mathrm{Supp}(w_i)^c) \cup \partial\,\mathrm{Supp}(w_i) \cup (\Sigma(P_{w_i}) \cap \mathrm{Supp}(w_i))$$

for $i \in [1, s-1]$.

(iii) The generalized decomposition $\widehat{w}$ is called **admissible** if $P^{w_i} \cap G_{w_i} \subseteq P_{i+1}$ for all $i \in [1, s-1]$.

(iv) The generalized decomposition $\widehat{w}$ is called **good** if it is admissible and $P_i = P_{w_i \cdots w_s}$ for all $i \in [1, s-1]$.

In the following $\widehat{w} = (w_1, \ldots, w_s)$ will always denote a reduced admissible generalized decomposition. We can apply the previous construction inductively as follows. The variety $\widehat{X}_n(\widehat{w})$ is defined to be $X_{P^{w_n}}(w_n)$. For each $i \in [1, s-1]$ we define $\widehat{X}_i(\widehat{w}) = \widehat{Y}(w_i)$ for $Y = \widehat{X}_{i+1}(\widehat{w})$. Moreover, the parabolic subgroup $P_{\widehat{X}_i(\widehat{w})}$ is given by P_i. Finally, we define $\widehat{X}(\widehat{w}) := \widehat{X}_1(\widehat{w})$. In the same way as for the Bott-Samelson variety the multiplication map $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X_{P^w}(w)$ is birational.

Lemma 7.3. *The variety $\widehat{X}(\widehat{w})$ is a tower of locally trivial fibrations with fibers $X_{P^{w_i}}(w_i)$.*

Proof. This follows directly from Lemma 7.1. □

In the (co)minuscule case there is a nice condition for a reduced generalized decomposition to be good.

Definition 7.4. We call $\widehat{w} = (w_1, \ldots, w_s)$ **(co)minuscule** if w is (co)minuscule and w_i is (co)minuscule in G_{w_i} for all $i \in [1, s]$.

Lemma 7.5. *Let $\widehat{w} = (w_1, \ldots, w_s)$ be (co)minuscule. Then $\widehat{w}$ is good if and only if $P^{w_i} \cap G_{w_i} \subseteq P_{w_{i+1}\cdots w_s}$ and $\partial \operatorname{Supp}(w_i) \subseteq \Sigma(P_{w_i \cdots w_s})$ for all $i \in [1, s-1]$.*

Proof. Notice that w being (co)minuscule implies $w_i \cdots w_s$ to be (co)minuscule for all $i \in [1, s]$. If $\widehat{w}$ is good, then clearly $\partial \operatorname{Supp}(w_i) \subseteq \Sigma(P_{w_i \cdots w_s})$ holds for all $i \in [1, s-1]$. Therefore, we are done with both directions of the proof if the following implication holds.

$$\partial \operatorname{Supp}(w_i) \subseteq \Sigma(P_{w_i \cdots w_s}) \ \forall i \in [1, s-1] \Rightarrow P_i = P_{w_i \cdots w_s} \ \forall i \in [1, s]$$

We will prove this by descending induction on i. The case $i = s$ simply follows by definition.

Let $\alpha \in \Sigma(P_{w_i \cdots w_s})$. We showed previously that $\Sigma(P_v) = \beta(\operatorname{Holes}(Q_v))$ holds for all (co)minuscule elements $v \in W$. If $\alpha \in \operatorname{Supp}(w_i)$ holds, then α corresponds to a hole of Q_{w_i}, which implies $\alpha \in \Sigma(P_{w_i}) \cap \operatorname{Supp}(w_i) \subseteq \Sigma(P_i)$. For $\alpha \in \partial \operatorname{Supp}(w_i)$ we clearly have $\alpha \in \Sigma(P_i)$. Finally, if $\alpha \in \operatorname{Supp}(w_i)^c \cap \partial \operatorname{Supp}(w_i)^c$, then α corresponds to a hole of $Q_{w_{i+1}\cdots w_s}$. By induction we get

$$\alpha \in \Sigma(P_{w_{i+1}\cdots w_s}) \cap \operatorname{Supp}(w_i)^c = \Sigma(P_{i+1}) \cap \operatorname{Supp}(w_i)^c \subseteq \Sigma(P_i).$$

Let $\alpha \in \Sigma(P_i)$. If $\alpha \in \operatorname{Supp}(w_i)$, then α is a hole of Q_{w_i}, hence a hole of $Q_{w_i \cdots w_s}$. That implies $\alpha \in \Sigma(P_{w_i \cdots w_s})$. For $\alpha \in \partial \operatorname{Supp}(w_i)$ the hypothesis implies $\alpha \in \Sigma(P_{w_i \cdots w_s})$. If $\alpha \in \operatorname{Supp}(w_i)^c \cap \partial \operatorname{Supp}(w_i)^c$, then $\alpha \in \Sigma(P_{i+1}) = \Sigma(P_{w_{i+1}\cdots w_s})$ by induction. Therefore, α corresponds to a hole of $Q_{w_i \cdots w_s}$, which implies $\alpha \in \Sigma(P_{w_i \cdots w_s})$. ☐

Lemma 7.6 ([Per07], Lemma 5.6)**.** *Let $\widehat{w}$ be (co)minuscule and $i \in [1, s-1]$. Assume further that for all $\beta \in \operatorname{Supp}(w_i)$ and $\beta' \in \operatorname{Supp}(w_{i+1})$ the equality $\langle \beta^\vee, \beta' \rangle = 0$ holds. The decompositions $\widehat{w}'$ and $\widehat{w}''$ defined by*

$$w'_k = \begin{cases} w_k & , \text{if } k \in [1, s] \backslash \{i, i+1\} \\ w_{i+1} & , \text{if } k = i \\ w_i & , \text{if } k = i+1 \end{cases} \qquad , w''_k = \begin{cases} w_k & , \text{if } k \in [1, i-1] \\ w_i w_{i+1} & , \text{if } k = i \\ w_{k+1} & , \text{if } k \in [i+1, s-1] \end{cases}$$

are (co)minuscule reduced admissible generalized decomposition. There are isomorphisms $\widehat{X}(\widehat{w}) \cong \widehat{X}(\widehat{w}') \cong \widehat{X}(\widehat{w}'')$, which respects the multiplication map to $X_{P^w}(w)$.

Proof. The claim on the decompositions is a simple calculation using the definition of being admissible. For the second statement we have to deal with the following situation. Let A, B be parabolic subgroups of a semisimple group G and C, D be parabolic subgroups of a semisimple group G'. Assume that B and D act on a variety X such that their actions commute. Moreover, u is an element of the Weyl group of G and v is an element of the Weyl group of G'. Then we consider the variety $\overline{AuB} \times^B (CvD \times^D X)$, where B acts on $CvD \times^D X$ by acting on X. There are the following isomorphisms.

$$\begin{aligned} \overline{AuB} \times^B (\overline{CvD} \times^D X) &\cong \overline{(A \times C)uv(B \times D)} \times^{B \times D} X \\ &\cong \overline{CvD} \times^D (\overline{AuB} \times^B X) \end{aligned}$$

That concludes the proof. □

We prove now that the Bott-Samelson resolution factors through the variety $\widehat{X}(\widehat{w})$. Let $\widetilde{w_i} = (s_{1,i}, \dots, s_{r_i,i})$ be a reduced decomposition of w_i for all $i \in [1, s]$.

Lemma 7.7 ([Per07], Lemma 5.7)**.** *The expression* $w = (\prod_{k=1}^{i} \prod_{j=1}^{r_k} s_{j,k}) \prod_{k=i+1}^{n} w_k$ *is an admissible generalized reduced decomposition for all* $i \in [1, s]$. *We denote it by* $\widetilde{w}'_i = (w'_1, \dots, w'_N)$.

Proof. Clearly, $\widetilde{w}'_i$ is reduced. Because $\widetilde{w}$ is admissible we have $P^{w'_k} \cap G_{w'_k} \subseteq P_{k+1}$ for all $k \in [N-n+i, N-1]$. For $k \in [1, N-n+i-1]$ there is $\beta \in S$ such that $w'_k = s_\beta$ and $G_{w'_k}$ is the semisimple group of rank 1 containing U_β. That implies $P^{w'_k} \cap G_{w'_k} \subseteq B \subseteq P_{k+1}$. Therefore, the expression $\widetilde{w}'_i$ is admissible. □

For all $\beta \in S$ we have

$$X_B(s_\beta) = \overline{B s_\beta B}/B \cong \overline{(P_{s_\beta} \cap G_{s_\beta}) s_\beta (P^{s_\beta} \cap G_{s_\beta})}/(P^{s_\beta} \cap G_{s_\beta}).$$

Therefore, $\widehat{X}(\widehat{w}'_n)$ is the Bott-Samelson variety. We have a map $\pi_i : \widetilde{X}(\widetilde{w}'_i) \to \widetilde{X}(\widetilde{w}'_{i-1})$ induced by the multiplication map

$$\prod_{j=1}^{r_i} \overline{(P_{s_{j,i}} \cap G_{s_{j,i}}) s_{j,i} (P^{s_{j,i}} \cap G_{s_{j,i}})} \to \overline{(P_{w_i} \cap G_{w_i}) w_i (P^{w_i} \cap G_{w_i})}.$$

Moreover, we denote the morphism $\widehat{\pi}_1 \circ \dots \circ \widehat{\pi}_n : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ by $\widetilde{\pi}$.

Proposition 7.8 ([Per07], Proposition 5.9)**.** *The map* $\widetilde{\pi}$ *is birational and the Bott-Samelson resolution* $\pi : \widetilde{X}(\widetilde{w}) \to X_{P^w}(w)$ *factors through* $\widetilde{\pi}$.

Proof. The same proof as for the Bott-Samelson resolution shows that $\widetilde{\pi}$ is birational. The second statement follows because it doesn't matter in which order we multiply. □

We are going to generalize the description of the Bott-Samelson variety in Proposition 4.7 to $\widehat{X}(\widehat{w})$. The map $\phi : G \times^{P^{w_1}} \dots \times^{P^{w_{s-1}}} G/P^{w_s} \to \prod_{i=1}^{s} G/P^{w_i}$ given by $\overline{(g_1, \dots, g_s)} \mapsto (\overline{g_1}, \overline{g_1 g_2}, \dots, \overline{g_1 \cdots g_s})$ is an isomorphism. Indeed, the inverse map is given by $(\overline{x_1}, \dots, \overline{x_s}) \mapsto \overline{(g_1, g_1^{-1} g_2, \dots, g_{s-1}^{-1} g_s)}$. We identify $\widehat{X}(\widehat{w})$ with it's image under ϕ.

Definition 7.9. By $m(Q_w)$ we denote the set of minimal elements of Q_w. The set $m_{\widehat{w}}(Q_w)$ is the union $\bigcup_{i=1}^{s} m(Q_{w_i})$.

Let $\widehat{w}$ be (co)minuscule. Then for $m \in m(Q_{w_i})$ the equality $P^{w_i} = P^{\beta_m}$ holds. Without much difficulty we can see that the projection $\prod_{i=1}^{s} \prod_{j=1}^{r_i} G/P^{\beta_{i,j}} \to \prod_{i \in m_{\widehat{w}}(Q_w)} G/P^{\beta_i}$ indcues the map $\widetilde{\pi}$.

Definition 7.10. By $p_{\widehat{w}}(Q_w)$ we denote the set $\bigcup_{i=1}^{s} p(Q_{w_i})$.

Proposition 7.11 ([Per07], Corollary 5.11)**.** *Let $\widehat{w}$ be (co)minuscule.*

(i) Let $K \subseteq [1, r]$. The variety Z_K is not contracted by $\widetilde{\pi}$ if and only if for all $i \in [1, s]$ the subword of w_i obtained by removing s_{β_j} for all $j \in K$ is reduced.

(ii) The divisor class group of $\widehat{X}(\widehat{w})$ is the free abelian group generated be the classes $\widetilde{\pi}_*(\xi_j)$ for all $j \in p_{\widehat{w}}(Q_w)$.

(iii) The group of curves modulo linear equivalence $A_1(\widehat{X}(\widehat{w}))$ is the free abelian group generated by $\widetilde{\pi}_*([C_j])$ for all $j \in m_{\widehat{w}}(Q_w)$.

Proof. This proposition follows from the fact that $\widehat{X}(\widehat{w})$ is defined as a tower of locally trivial fibrations with fibers $X_{P^{w_i}}(w_i)$. Part (i) follows from Proposition 4.26. The other two statements follow from Theorem 3.8. The fact that the decomposition is (co)minuscule is only needed for the concrete description as images of $\widetilde{\pi}$. □

7.2 Decompositions of Cominuscule Quivers

Keeping the notation of last chapter, we want to find cominuscule good generalized reduced decompositions. Let G be a group with root system C_n and P the maximal parabolic subgroup with $\Sigma(P) = \{\alpha_n\}$, where α_n is the long simple root (see [Bou68] notations). All the statements in this section have corresponding results in the simply laced minuscule case due to [Per07]. Let $w \in W$ be a cominuscule element.

Definition 7.12. (i) Let $q \in Q_w$ be the biggest vertex with $\beta_q = \alpha_n$. We define

$$p'(Q_w) = \begin{cases} p(Q_w) \cup \{q\} & \text{, if } \alpha_{n-1} \in \beta(p(Q_w)) \\ p(Q_w) & \text{, otherwise.} \end{cases}$$

(ii) Let $A \subseteq p'(Q_w)$. Then we define the set

$$\widehat{Q}_w(A) = \{i \in Q_w \mid \exists j \in p'(Q_w)\backslash A,\ i \preceq j\}.$$

The complement is denoted by $Q_w(A) := Q_w \backslash \widehat{Q}_w(A)$.

We show that these subquivers correspond again to cominuscule elements.

Proposition 7.13. *Let $A \subseteq p(Q_w) \subset p'(Q_w)$.*

(i) If $\#A = 1$, then $Q_w(A)$ is connected.

(ii) The subquiver $\widehat{Q}_w(A)$ corresponds to a cominuscule element.

(iii) The equalities $p'(Q_w(A)) = p(Q_w(A)) = A$ and $p'(\widehat{Q}_w(A)) = p'(Q_w)\backslash A$ hold.

(iv) Each connected component C of $Q_w(A)$ corresponds to a cominuscule element in the Weyl group of the semisimple group containing U_α for all $\alpha \in \beta(C)$.

Proof. The first and third statement are obvious. The second claim follows directly from Theorem 5.9. We are left to prove the fourth statement. This will be done in several steps. First we prove that C has a unique minimal element. This can be used to show that C fulfills the properties of Lemma 5.14.

Assume there are two different minimal elements $j_1 \neq j_2$ of C. Let $j_1 = i_0, i_1, \ldots, i_l = j_2$ be a sequence of vertices in C such that for all $k \in [0, l-1]$ there is an arrow connecting i_k and i_{k+1}. We can choose this sequence such that l is minimal. Moreover, let $x \in [1, l-1]$ be minimal such that $i_x \succeq i_{x-1}$ and $i_x \succeq i_{x+1}$. Such an x exists because j_1 and j_2 are both minimal elements. By the minimality of l we also have $i_{x-1} \neq i_{x+1}$. Therefore, Lemma 5.14 yields the existence of $s(i_x)$ as an element of Q_w. But by Proposition 6.1 there is no other vertex than i_{x-1} and i_{x+1} in Q_w with an arrow to $s(i_x)$. That means $s(i_x)$ is in C. Replacing i_x in the sequence by $s(i_x)$ we get a new sequence where x is reduced by 1. Repeating this process we get a sequence with $i_1 \succeq i_0$ and $i_1 \succeq i_2$. By the same argument we have $s(i_1) \in C$ and get a contradiction to j_1 being minimal. Hence, there is a unique minimal element in C.

Next we prove the three properties of Lemma 5.14.

(i) We need to prove that the unique minimal vertex in C corresponds to the simple root of a cominuscule fundamental weight in the Weyl group of the semisimple group containing U_α for $\alpha \in \beta(C)$. Let m be the minimal vertex in C and α_n the long simple root. Since in the root system of type A_n all fundamental weights are cominuscule and in the C_n case only the fundamental weight corresponding to α_n is cominuscule, we need to prove the following implication.

$$\beta_m \neq \alpha_n \Rightarrow \alpha_n \notin \beta(C)$$

Let $\beta_m \neq \alpha_n$ and assume $\alpha_n \in \beta(C)$. Then we can choose $k \in C$ minimal such that $\beta_k = \alpha_n$. By Proposition 6.1 there is a unique vertex $j \in Q_w$ with an arrow coming from k. Due to k not being minimal we get $j \in C$. Again by Proposition 6.1 the only arrow ending in $s(k)$ comes from j. But that implies $s(k) \in C$ contradicting the minimality of k.

(ii) Let $k \in C$ be any element which is not minimal. Assume $s(k)$ doesn't exist in C. Then there are two possibilities. On the one hand, assume that $s(k)$ doesn't exist in Q_w. By Lemma 5.14 there is a unique arrow from k to a vertex $j \in Q_w$. Because k is not minimal we get $j \in C$. On the other hand, assume that $s(k)$ does exists, but is not a vertex of C. Again using Proposition 5.14 there are at most two vertices $k_1, k_2 \in Q_w$ with an arrow coming from k. Since k is not minimal one of k_1 and k_2 has to be in C. But these are the only vertices with an arrow to $s(k)$.

Since $s(k)$ is not an element of C, we get $k_1 \neq k_2$ and $k_1 \notin C$ or $k_2 \notin C$. But that means exactly one of them is in C.

(iii) Let $k \in C$ be any element which is not minimal. Assume $s(k)$ exists as an element of C. Again there are at most two different vertices k_1 and k_2 between k and $s(k)$. If $k_1 \notin C$ or $k_2 \notin C$, then $s(k) \notin C$ which contradicts the assumption.

Thus, C corresponds to a cominuscule element. □

Having this proposition in mind we are able to decompose the quiver such that we get a cominuscule good generalized reduced decomposition. Let $(A_i)_{i\in[1,s]}$ be a partition of $p'(Q_w)$. If there is a vertex $p \in p(Q_w)$ with $\beta_p = \alpha_{n-1}$ and q is the biggest vertex such that $\beta_q = \alpha_n$, then we assume the condition

$$p \in A_k,\ q \in A_l \Rightarrow k < l. \tag{7.1}$$

Inductively, we define $Q_0 := Q_w$ and $Q_i := \widehat{Q}_{i-1}(A_i)$ for all $i \in [1, s]$. Note, that this depends on the ordering of the sets A_i.

Lemma 7.14. *By setting $Q_{w_i} := Q_{i-1}(A_i)$ we get a cominuscule good generalized reduced decomposition $\widehat{w} = (w_1, \dots, w_s)$ of w. Moreover, the equality $p'(Q_w) = p_{\widehat{w}}(Q_w)$ holds.*

Proof. Except from the fact that $\widehat{w}$ is good everything follows easily from the last proposition. To show that $\widehat{w}$ is good we will use Lemma 7.5.

(i) At first, we show the inclusion $P^{w_i} \cap G_{w_i} \subseteq P_{w_{i+1}\cdots w_s}$. This follows from the following implication.

$$j \in \text{Holes}(\widehat{Q}_{i-1}(A_i)),\ \beta_j \in \text{Supp}(w_i) \Rightarrow\ p(j) \text{ exists and is minimal in } Q_{i-1}(A)$$

The existence of $p(j)$ follows from $\beta_j \in \text{Supp}(w_i)$. There are one or two vertices between j and $p(j)$. Due to j being a hole, they are both in $\widehat{Q}_{i-1}(A)$. But that means no element in $Q_{i-1}(A)$ is smaller than $p(j)$.

(ii) To conclude the proof we need to show the inclusion $\partial\,\text{Supp}(w_i) \subseteq \Sigma(P_{w_i\cdots w_s})$. Let $\alpha \in \partial\,\text{Supp}(w_i)$. If we have $\alpha \notin \text{Supp}(w_i \cdots w_s)$, then α corresponds to a virtual hole and we are done. If $\alpha \in \text{Supp}(w_i \cdots w_s)$, then there is a vertex $j \in Q_{w_i\cdots w_s}$ such that $\beta_j = \alpha$. We choose j to be the biggest such vertex. Because $\beta_j \in \partial\,\text{Supp}(w_i)$, there is a vertex $k \in Q_{w_i}$ with an arrow to j. The maximality of j implies that $p(j)$ does not exist. Assume that j is not a hole of $Q_{w_i\cdots w_s}$. Then k is the only vertex with an arrow to j and we get $j \in Q_{w_i}$. That is a contradiction to $\beta_j \in \partial\,\text{Supp}(w_i)$. Therefore, we have $\alpha = \beta_j \in \Sigma(P_{w_i\cdots w_s})$.

Therefore, $\widehat{w}$ is indeed good. □

We want to investigate further into some special cases of the last construction.

Definition 7.15. We say that a partition $(A_i)_{i\in[1,s]}$ of $p'(Q_w)$ satisfying (7.1) is of **construction 1** if $\#A_i = 1$ for all $i \in [1,s]$.

Before we can introduce further constructions we need to prove some other results first.

Lemma 7.16. *Let $\widehat{w}$ be obtained from construction 1. Assume that $j,k \in m_{\widehat{w}}(Q_w)$ are vertices such that there is an $x \in Q_w$ with $x \succeq j$ and $x \succeq k$. Then we have either $j \succeq k$ or $k \preceq j$.*

Proof. There are $a,b \in [1,s]$ such that $j \in Q_{w_a}$ and $k \in Q_{w_b}$. We prove the lemma by induction on (a,b) via the following ordering on $[1,s]^2$. The relation $(i,j) \leq (k,l)$ holds if $\max\{i,j\} < \max\{k,l\}$ or the two inequalities $\max\{i,j\} \leq \max\{k,l\}$ and $|i-j| \leq |k-l|$ hold. If the equality $a = b$ holds, then we have $j = k$.

Assume $a \neq b$ holds. Let $x \in Q_w$ be a minimal vertex for the condition $x \succeq j$ and $x \succeq k$. Assume $j \neq x \neq i$. By Lemma 5.14 there are in between one or two arrows starting in x. If there would be a unique arrow starting in x, we get a contradiction to the minimality of x. Therefore, there are two arrows starting in x ending in some vertices $y_1, y_2 \in Q_w$. Again using the minimality of x we can assume without loss of generality that we have the relations $y_1 \succeq j$, $y_1 \not\succeq k$, $y_2 \not\succeq j$ and $y_2 \succeq k$. Because there are two arrows starting in x, the successor $s(x)$ exists. If the relation $s(x) \succeq j$ holds, then we get the contradiction $y_2 \succeq j$. Therefore, the relation $s(x) \not\succeq j$ holds. The same argument yields $s(x) \not\succeq k$.

There are $c,d,e \in [1,s]$ such that $s(x) \in Q_{w_c}$, $y_1 \in Q_{w_d}$ and $y_2 \in Q_{w_e}$. Due to $s(x) \prec y_1, y_2$ the inequalities $c \geq d$ and $c \geq e$ hold. Similarly, we have $a \geq d$ and $b \geq e$. Because $\widehat{w}$ is of construction 1, there is a unique peak p of Q_{w_c}. Therefore, we have the relation $s(x) \preceq p$. Since the only vertices with arrows to $s(x)$ are y_1 and y_2 (see Proposition 6.1), we also have $p \succeq y_1$ or $p \succeq y_2$. That implies $c \leq d$ or $c \leq e$. Hence, we established $c = d$ or $c = e$. Without loss of generality we may assume $c = d$.

At this point we distinguish between several cases.

(i) Assume the equalities $c = d = a$ hold. The vertex j is minimal in Q_{w_a} which gives the contradiction $y_2 \succeq s(x) \succeq j$.

(ii) Assume $c = d < a < b$. Let m be the minimal vertex in Q_{w_c}. Then we have the relations $x \succ s(x) \succeq m$ and $x \succeq j$. We use induction ($(c,a) < (a,b)$) to get $m \preceq j$ or $m \succeq j$. But now $c < a$ implies $m \succeq j$. That gives the contradiction $y_2 \succeq s(x) \succeq m \succeq j$.

(iii) Assume $b < c = d < a$. Then we have $(c,a) < (a,b)$. Therefore, we can do the same as in the second case.

(iv) Assume $c = d \leq b < a$. Then we have the relations $x \succ s(x) \succeq m$ and $x \succeq k$. We use induction ($(c,b) < (a,b)$) to get $m \preceq k$ or $m \succeq k$. But now $c < b$ implies $m \succeq k$. That gives the contradiction $y_1 \succeq s(x) \succeq m \succeq k$.

These are all possible cases which concludes the proof. □

Corollary 7.17. *Let $\widehat{w}$ be obtained from construction 1 and $i \in m_{\widehat{w}}(Q_w)\backslash m(Q_w)$. Then the element*

$$f(i) := \max\{j \in m_{\widehat{w}}(Q_w) \mid i \succ j\}$$

exists.

Proof. The preceding lemma shows that $\{j \in m_{\widehat{w}}(Q_w) \mid i \succ j\}$ is totally ordered. The finiteness of $m_{\widehat{w}}(Q_w)$ proves the lemma. □

In construction 1 every quiver Q_{w_i} has a unique peak. Therefore, we define f on the peaks as follows. For $p \in p(Q_{w_i})$ let m be the corresponding minimal element Q_{w_i}. Then $f(p)$ is defined to be the peak above $f(m)$. With this in mind we are able to give two new constructions as a special case of construction 1.

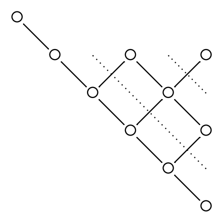
The following idea of a neat ordering was first introduced in [Zel83].

Definition 7.18. (i) An ordering $(i_1, \ldots, i_s)$ on the set of peaks of Q_w is called **neat** if for all $k \in [1, s-1]$ the inequality $h(i_k) \leq h(f(i_k))$ holds. A partition $(A_i)_{i\in[1,s]}$ is called of **construction 2** if it is obtained from a neat ordering using construction 1.

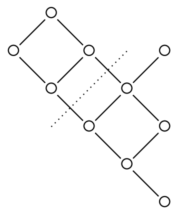
(ii) An ordering $(i_1, \ldots, i_s)$ on the set of peaks of Q_w is called **kind of neat** if for all $k \in [1, s-1]$ the inequality $h'(i_k) \leq h'(f(i_k))$ holds. A partition $(A_i)_{i\in[1,s]}$ is called of **construction 3** if it is obtained from a kind of neat ordering using construction 1.

Let us give some examples of these decompositions.

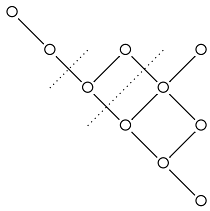
Example 7.19. (i) The following decomposition in the C_6 case is of construction 2, but not 3.



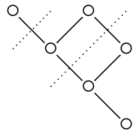
(ii) This decomposition in the C_5 case is of construction 3, but not 2.



(iii) The third decomposition in the C_6 case is of construction 1, but neither 2 nor 3.



(iv) This last decomposition in the C_4 case is of construction 3, but not 2.



8 Relative Mori Theory of Cominuscule Schubert Varieties

In this chapter we describe all relative minimal models of cominuscule Schubert varieties in the C_n and B_n case. This will be done by mimicking the approach for minuscule Schubert varieties in [Per07]. As Mori theory is only developed in characteristic zero, we need to assume $k = \mathbb{C}$. For more background on Mori theory we refer to [Mat02]. After establishing some general results we will deal with the C_n and B_n cases separately.

Let $\widehat{w}$ be a cominuscule good reduced generalized decomposition of an element $w \in W$. Recall from previous chapters that we have the following commutative diagram.

$$\begin{array}{ccc} \widetilde{X}(\widetilde{w}) & \xrightarrow{\widetilde{\pi}} & \widehat{X}(\widehat{w}) \\ & \searrow^{\pi} & \downarrow{\scriptstyle \widehat{\pi}} \\ & & X_{P^w}(w) \end{array}$$

For all $i \in p_{\widehat{w}}(Q_w)$ we denote $\widehat{D}_i := \widetilde{\pi}_*(\xi_i)$. In Proposition 7.11 we proved the isomorphisms

$$A^1(\widehat{X}(\widehat{w})) \cong \bigoplus_{i \in p_{\widehat{w}}(Q_w)} \mathbb{Z} \cdot \widehat{D}_i$$

and

$$A_1(\widehat{X}(\widehat{w})) \cong \bigoplus_{i \in m_{\widehat{w}}(Q_w)} \mathbb{Z} \cdot \widetilde{\pi}_*[C_i].$$

We have also seen that $\widehat{X}(\widehat{w})$ can be identified with a subvariety of $\prod_{i \in m_{\widehat{w}}(Q_w)} G/P^{\beta_i}$. Therefore, there is a map $p_i : \widehat{X}(\widehat{w}) \to G/P^{\beta_i}$ induced by the corresponding projection for all $i \in m_{\widehat{w}}(Q_w)$. We define the sheaf $\mathcal{M}_i$ as $p_i^*\mathcal{O}(1)$. By the definition of the sheaf $\mathcal{L}_i$ on the Bott-Samelson variety $\widetilde{X}(\widetilde{w})$ it is not hard to see that $\widetilde{\pi}^*\mathcal{M}_i = \mathcal{L}_i$ holds.

Lemma 8.1. *A basis of* $\mathrm{Pic}(\widehat{X}(\widehat{w}))$ *is given by* $(\mathcal{M}_i)_{i \in m_{\widehat{w}}(Q_w)}$.

Proof. The variety $\widehat{X}(\widehat{w})$ has been defined via a tower of locally trivial fibration with fibers being cominuscule Schubert varieties. These Schubert varieties have a Picard group isomorphic to $\mathbb{Z}$. The $\mathcal{M}_i$ correspond to the generators of them. □

A dual basis for curves can be given as follows.

Lemma 8.2. *A basis of* $A_1(\widehat{X}(\widehat{w}))$ *dual to* $(\mathcal{M}_i)_{i\in m_{\widehat{w}}(Q_w)}$ *is given by* $(\widetilde{\pi}_*(Y_i))_{i\in m_{\widehat{w}}(Q_w)}$.

Proof. It suffices to show the duality. We have the equalities

$$\begin{aligned}\widetilde{\pi}_*([Y_i])\cdot\mathcal{M}_j &= \widetilde{\pi}_*([Y_i]\cdot\widetilde{\pi}^*\mathcal{M}_j)\\ &= \widetilde{\pi}_*([Y_i]\cdot\mathcal{L}_j).\end{aligned}$$

Indeed, the corresponding result on the Bott-Samelson variety concludes the lemma. □

Proposition 8.3. *A basis of the cone of effective curves of the variety* $\widehat{X}(\widehat{w})$ *is given by* $(\widetilde{\pi}_*(Y_i))_{i\in m_{\widehat{w}}(Q_w)}$. *A basis of the closure of the ample cone is given by* $(\mathcal{M}_i)_{i\in m_{\widehat{w}}(Q_w)}$.

Proof. On the one hand, the embedding $\widehat{X}(\widehat{w})\longhookrightarrow\prod_{i\in m_{\widehat{w}}(Q_w)}G/P^{\beta_i}$ is induced by $\bigotimes_{i\in m_{\widehat{w}}(Q_w)}\mathcal{M}_i$. Therefore, the cone generated by the $\mathcal{M}_i$ is contained in the closure of the ample cone.

On the other hand, let A be any ample divisor on $\widehat{X}(\widehat{w})$. Then we have $a_i := A\cdot\widetilde{\pi}_*[Y_i] > 0$ for all $i\in m_{\widehat{w}}(Q_w)$. The divisor $A-\sum_{i\in m_{\widehat{w}}(Q_w)}a_i\mathcal{M}_i$ is numerically trivial. Therefore, A is in the cone generated by the $\mathcal{M}_i$.

Duality gives the result on curves. □

8.1 The C_n-Case

Let G be a semisimple linear algebraic group with root system of type C_n. By P we denote the maximal parabolic subgroup corresponding to the unique longest simple root α_n. In this whole section $\widehat{w}=(w_1,\dots,w_s)$ is a generalized reduced expression of construction 1. The goal is to prove that the relative minimal models of a Schubert variety $X_P(w)$ are exactly given by the varieties $\widehat{X}(\widehat{w})$, where $\widehat{w}$ is obtained by construction 3. In order to do this we will first give more explicit formulas for the involved sheaves and second do the needed computations.

Proposition 8.4. *Let* $i\in m_{\widehat{w}}(Q_w)$. *Then we have the equality*

$$\mathcal{M}_i=\begin{cases}\sum_{\substack{k\in p_{\widehat{w}}(Q_w)\\ k\succeq i,\beta_k\neq\alpha_n}}2\widehat{D}_k+\sum_{\substack{k\in p_{\widehat{w}}(Q_w)\\ k\succeq i,\beta_k=\alpha_n}}\widehat{D}_k & \text{, if } \beta_i=\alpha_n\\ \sum_{\substack{k\in p_{\widehat{w}}(Q_w)\\ k\succeq i}}\widehat{D}_k & \text{, otherwise.}\end{cases}$$

Proof. Due to $\widehat{X}(\widehat{w})$ being normal we get the equality $\widetilde{\pi}_*\mathcal{O}_{\widetilde{X}(\widehat{w})}=\mathcal{O}_{\widehat{X}(\widehat{w})}$. The projection formula yields $\widetilde{\pi}_*\mathcal{L}_i=\mathcal{M}_i$. We define the set $A_i:=\{k\in p_{\widehat{w}}(Q_w)\mid k\succeq i\}$. By Proposition 7.13 the subquiver $Q_w(A_i)$ is cominuscule with unique minimal element i. For any vertices $j,k\in Q_w$ with neither $k\not\succeq i$ nor $k\not\preceq i$ the equality $\langle\gamma_k^\vee,\gamma_i\rangle=0$ holds by the propositions 6.2 and 6.21. Therefore, we can calculate $\mathcal{M}_i$ using Proposition 4.36 by only looking at $Q_w(A_i)$. The result follows from Proposition 6.4 if $\beta_i = \alpha_n$ and Proposition 6.23 otherwise. □

Because $\widehat{X}(\widehat{w})$ is defined via a tower of locally trivial fibration we get the following lemma.

Lemma 8.5. *The variety $\widehat{X}(\widehat{w})$ is locally ($\mathbb{Q}$-)factorial if and only if $X_{P^{w_i}}(w_i)$ is locally ($\mathbb{Q}$-)factorial. In particular, $\widehat{X}(\widehat{w})$ is locally $\mathbb{Q}$-factorial if $\widehat{w}$ is obtained from construction 1.*

We want to compute the canonical divisor of $\widehat{X}(\widehat{w})$. We need the following proposition in order to do this.

Proposition 8.6. *The morphism $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ is rational.*

Proof. In Section 7.1 we decomposed the map $\widetilde{\pi}$ into a sequence of maps π_i that are induced by Bott-Samelson resolutions on the fiber. The result follows because the Bott-Samelson resolution is rational. □

Using Corollary 6.5 we get the following corollary.

Corollary 8.7. *We have the equality*

$$-2K_{\widehat{X}(\widehat{w})} = \sum_{i \in p_{\widehat{w}}(Q_w), \beta_i = \alpha_n} (h'(i) + 2)\widehat{D}_i + \sum_{i \in p_{\widehat{w}}(Q_w), \beta_i \neq \alpha_n} 2(h'(i) + 2)\widehat{D}_i.$$

Since every quiver Q_{w_i} has a unique peak p_i, we write $h'(w_i) := h'(p_i)$. We can give a description of $K_{\widehat{X}(\widehat{w})}$ in terms of the $\mathcal{M}_i$. Recall that we have the function $f : m_{\widehat{w}}(Q_w) \backslash m(Q_w) \to m_{\widehat{w}}(Q_w)$ given by $i \mapsto \max\{j \in m_{\widehat{w}}(Q_w) \mid i \succ j\}$.

Corollary 8.8. *We have the equality*

$$-2K_{\widehat{X}(\widehat{w})} = \sum_{i \in m_{\widehat{w}}, \beta_i = \alpha_n} (h'(w_i) - h'(w_{f(i)}))\mathcal{M}_i + \sum_{i \in m_{\widehat{w}}, \beta_i \neq \alpha_n} 2(h'(w_i) - h'(w_{f(i)}))\mathcal{M}_i,$$

where $h'(w_{f(i)}) = -2$ if $f(i)$ is not defined. In particular, $\widehat{X}(\widehat{w})$ is $\mathbb{Q}$-Gorenstein.

Proof. This is just an induction on $s = \# m_{\widehat{w}}(Q_w)$. □

Notice, that the fact that $\widehat{X}(\widehat{w})$ is $\mathbb{Q}$-Gorenstein was already known because it is locally $\mathbb{Q}$-factorial. This corollary enables us to prove the next proposition. We refer to [Mat02] for definition of terminal singularities.

Proposition 8.9. *The variety $\widehat{X}(\widehat{w})$ has at most terminal singularities.*

Proof. For any $i \in Q_w$ we define

$$\lambda_i := \begin{cases} 1 & \text{, if } \beta_i = \alpha_n \\ 2 & \text{, if } \beta_i \neq \alpha_n. \end{cases}$$

We have the equality

$$-2\widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i \in m_{\widehat{w}}} \lambda_i (h'(w_i) - h'(w_{f(i)}))\mathcal{L}_i.$$

Moreover, we can use the same argumentation as in Proposition 8.4 to get the equality

$$\mathcal{L}_i = \begin{cases} \sum_{k \succeq i} \lambda_i \xi_k & \text{, if } \beta_i = \alpha_n \\ \sum_{k \succeq i} \xi_k & \text{, otherwise.} \end{cases}$$

Therefore, we can compute

$$\begin{aligned} -2\widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} &= \sum_{i \in m_{\widehat{w}(Q_w)}} \sum_{k \succeq i} \lambda_k (h'(w_i) - h'(w_{f(i)}))\xi_k \\ &= \sum_{i \in m_{\widehat{w}(Q_w)}} \sum_{k \in Q_{w_i}} \lambda_k (h'(w_i) + 2)\xi_k. \end{aligned}$$

Subtracting yields the equality

$$2(K_{\widetilde{X}(\widetilde{w})} - \widetilde{\pi}^* K_{\widehat{X}(\widehat{w})}) = \sum_{i \in m_{\widehat{w}(Q_w)}} \sum_{k \in Q_{w_i}} \lambda_k (h'(w_i) - h'(k))\xi_k.$$

We have the inequality $h'(w_i) - h'(k) \geq 0$ with equality if and only if $k \in Q_{w_i}$ is the peak of Q_{w_i}. The proposition follows because ξ_k is contracted if and only if $k \notin p_{\widehat{w}}(Q_w)$. □

At this point we can compute some relative minimal models of $X_P(w)$. Later we show that they are indeed all relative minimal models in the same birational class as $X_P(w)$. Recall from [Mat02] that a projective variety Y with a morphism $\varphi : Y \to X_P(w)$ is called a relative minimal model of $X_P(w)$ if Y is normal with only $\mathbb{Q}$-factorial and terminal singularities and the canonical divisor K_Y is φ-nef, i.e. $[K_Y] \cdot [C] \geq 0$ for all curves $C \subseteq Y$ contracted by φ.

Theorem 8.10. *If $\widehat{w}$ is obtained from construction 3, then $\widehat{X}(\widehat{w})$ is a relative minimal model of $X_P(w)$.*

Proof. The only condition which is left to prove is that $K_{\widehat{X}(\widehat{w})}$ is $\widehat{\pi}$-nef. Using Proposition 8.3 we only need to deal with the case $C = \widetilde{\pi}_*[Y_i]$ for some $i \in m_{\widehat{w}}(Q_w)$. We have the equality

$$[K_{\widehat{X}(\widehat{w})}] \cdot \widetilde{\pi}_*[Y_i] = \begin{cases} h'(w_{f(i)}) - h'(w_i) & \text{, if } \beta_i = \alpha_n \\ 2(h'(w_{f(i)}) - h'(w_i)) & \text{, if } \beta_i \neq \alpha_n. \end{cases}$$

This is greater or equal to zero for all $i \in m_{\widehat{w}}(Q_w) \backslash m(Q_w)$, where $\widetilde{\pi}_*[Y_i]$ is contracted. □

We have proven that the cone of effective curves modulo numerical equivalence $\overline{\mathrm{NE}}(\widehat{X}(\widehat{w}))$ is generated by $(\widetilde{\pi}_*[Y_i])_{i\in m_{\widehat{w}}(Q_w)}$. Recall that the cone theorem ([Mat02] Theorem 7-2-1) says

$$\overline{\mathrm{NE}}(\widehat{X}(\widehat{w})) = \{z \in \overline{\mathrm{NE}}(\widehat{X}(\widehat{w})) \mid K_{\widehat{X}(\widehat{w})} \cdot z \geq 0\} + \sum R_l,$$

where the R_l are discrete in

$$\{z \in \overline{\mathrm{NE}}(\widehat{X}(\widehat{w})) \mid K_{\widehat{X}(\widehat{w})} \cdot z < 0\}$$

and $R_l = \mathbb{R}_+[l]$ for some curve $l \subseteq \widehat{X}(\widehat{w})$. The R_l are called extremal rays. Replacing $K_{\widehat{X}(\widehat{w})}$ by $K_{\widehat{X}(\widehat{w})} + \epsilon D$ for any effective $\mathbb{Q}$-Cartier divisor D and $0 < \epsilon << 1$ gives the definition of $(K_{\widehat{X}(\widehat{w})} + \epsilon D)$-extremal rays. Our next step is to compute these rays.

Lemma 8.11. *Let D be any effective divisor of $\widehat{X}(\widehat{w})$ that is $\mathbb{Q}$-Cartier. Then the $(K_{\widehat{X}(\widehat{w})} + \epsilon D)$-extremal rays are generated by the classes $\widetilde{\pi}_*[Y_j]$ for $j \in m_{\widehat{w}}(Q_w)$ such that $(K_{\widehat{X}(\widehat{w})} + \epsilon D) \cdot \widetilde{\pi}_*[Y_j] < 0$.*

Proof. Let $[C] = \sum_{i\in m_{\widehat{w}}(Q_w)} a_i \widetilde{\pi}_*[Y_i]$ be any effective divisor. We denote by μ_j (resp. ν_k, ω_l) the $\widetilde{\pi}_*[Y_i]$ such that $(K_{\widehat{X}(\widehat{w})} + \epsilon D) \cdot \widetilde{\pi}_*[Y_j] < 0$ (resp. $(K_{\widehat{X}(\widehat{w})} + \epsilon D) \cdot \widetilde{\pi}_*[Y_j] > 0$, $(K_{\widehat{X}(\widehat{w})} + \epsilon D) \cdot \widetilde{\pi}_*[Y_j] = 0$). For all j, k we have positive coefficients $x_{j,k}, y_{j,k}$ such that $(K_{\widehat{X}(\widehat{w})} + \epsilon D) \cdot (x_{j,k}\mu_j + y_{j,k}\nu_k) = 0$. Assume that $(K_{\widehat{X}(\widehat{w})} + \epsilon D) \cdot [C] < 0$ holds. It is not hard to see that $[C]$ has to be a linear combination with positive coefficients of the classes (μ_j), $(x_{j,k}\mu_j + y_{j,k}\nu_k)$ and (ω_l). Therefore, the classes (μ_j) are the $(K_{\widehat{X}(\widehat{w})} + \epsilon D)$-extremal rays. □

The next step is to compute all the relative flops of the minimal models computed in Theorem 8.10 (see again [Mat02] for a definition). Let $(p_1, \ldots, p_s)$ be a kind of neat ordering of $p'(Q_w)$. Assume that the generalized decomposition $\widehat{w}$ is obtained from this ordering via construction 3. For any $i \in [1, s-1]$ with $\beta_{p_i} \neq \alpha_{n-1}$ or $\beta_{p_{i+1}} \neq \alpha_n$ we define $q_k = p_k$ for $k \in [1, s]\backslash\{i, i+1\}$, $q_i = p_{i+1}$ and $q_{i+1} = p_i$. Let $\widehat{w}' = (w'_1, \ldots, w'_s)$ be the generalized reduced decomposition obtained from construction 1 via the ordering $(q_1, \ldots, q_s)$. A third decomposition $\widehat{w}'' = (w''_1, \ldots, w''_{s-1})$ shall be given by $w''_k = w_k$ for $k \in [1, i-1]$, $w''_i = w_i w_{i+1}$ and $w''_k = w_{k+1}$ for $k \in [i+1, s-1]$. That means $\widehat{w}''$ is obtained by the partition $(A_k)_{k\in[1,s-1]}$ of the peaks of Q_w given by $A_k = \{p_k\}$ for $k \in [1, i-1]$, $A_i = \{i, i+1\}$ and $A_k = \{k+1\}$ for $k \in [i+1, s-1]$. We have morphisms

$$\varphi : \widehat{X}(\widehat{w}) \to \widehat{X}(\widehat{w}''),\ \varphi' : \widehat{X}(\widehat{w}') \to \widehat{X}(\widehat{w}'')$$

induced by multiplication. For the definition of flops see again [Mat02].

Proposition 8.12. *There are elements $k_i \in m(Q_{w_i})$, $k'_i \in m(Q_{w'_i})$, $k_{i+1} \in m(Q_{w_{i+1}})$ and $k'_{i+1} \in m(Q_{w'_{i+1}})$. Moreover, let $D := \widehat{D}_{p_{i+1}}$ and $D' := \widehat{D}'_{q_i}$.*

(i) If $k_i \not\preceq k_{i+1}$ (i.e. $f(i) \neq i+1$), then φ and φ' are isomorphisms.

(ii) If $k_i \succeq k_{i+1}$ *(i.e.* $f(i) = i+1$*) and* $K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_{k_i}] = 0$*, then* φ *(resp.* φ'*) is the contraction corresponding to the* $(K_{\widehat{X}(\widehat{w})} + \epsilon D)$*-extremal ray* $\mathbb{R}_+\widetilde{\pi}_*[Y_{k_i}]$ *(resp. the* $(K_{\widehat{X}(\widehat{w}')} + \epsilon D')$*-extremal ray* $\mathbb{R}_+\widetilde{\pi}_*[Y_{k_i'}]$*). Moreover,* (φ', D') *is the relative flop of* (φ, D)*.*

Proof. The hypothesis of the first statement yields the hypothesis of Lemma 7.6. Therefore, φ and φ' are isomorphisms.

In the second case, we know that $\widehat{X}(\widehat{w})$ and $\widehat{X}(\widehat{w}')$ are normal, projective varieties with only $\mathbb{Q}$-factorial and terminal singularities. Moreover, the morphisms φ and φ' are birational onto the normal, projective variety $X_P(w)$.

Descending induction on the ordering of $m_{\widehat{w}}(Q_w)$ together with Proposition 8.4 yields $\widetilde{\pi}_*[Y_{k_i}] \cdot \widehat{D}_{p_i} > 0$, $\widetilde{\pi}_*[Y_{k_i}] \cdot \widehat{D}_{p_{i+1}} < 0$ and $\widetilde{\pi}_*[Y_{k_i}] \cdot \widehat{D}_{p_k} = 0$ for all $k \in m_{\widehat{w}}(Q_w)\backslash\{k_i\}$ with $k \succ k_{i+1}$. Therefore, $\widetilde{\pi}_*[Y_{k_i}]$ is extremal for $K_{\widehat{X}(\widehat{w})} + \epsilon D$.

The group of Weil divisors of $\widehat{X}(\widehat{w})$ and $\widehat{X}(\widehat{w}')$ has a basis indexed by $p'(Q_w)$. Because $\beta_{p_i} \neq \alpha_{n-1}$ or $\beta_{p_{i+1}} \neq \alpha_n$ that is also the case for $\widehat{X}(\widehat{w}'')$. Thus no divisors are contracted by φ or φ' and both morphisms are Mori-small. By our description of the Picard group in Lemma 8.1 the relative Picard numbers are $\rho(\widehat{X}(\widehat{w})/\widehat{X}(\widehat{w}'')) = \rho(\widehat{X}(\widehat{w}')/\widehat{X}(\widehat{w}'')) = 1$.

By assumption we have the equality $K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_{k_i}] = 0$. Therefore, $K_{\widehat{X}(\widehat{w})}$ is φ-trivial. That also implies the equality $h'(w_i) = h'(w_{i+1})$, which shows $K_{\widehat{X}(\widehat{w}')} \cdot \widetilde{\pi}_*[Y_{k_i'}] = 0$. Thus $K_{\widehat{X}(\widehat{w}')}$ is φ'-trivial. We already showed $D \cdot \widetilde{\pi}_*[Y_{k_i}] < 0$, which means $-D$ is φ-ample. We have the following commutative diagram.

$$\begin{array}{ccc} \widetilde{X}(\widetilde{w}) & \xrightarrow{\widetilde{\pi}} & \widehat{X}(\widehat{w}) \\ \downarrow{\scriptstyle \widetilde{\pi}'} & & \downarrow{\scriptstyle \varphi} \\ \widehat{X}(\widehat{w}') & \xrightarrow{\varphi'} & \widehat{X}(\widehat{w}'') \end{array}$$

Thus, the strict transform of D is given by $\widetilde{\pi}'_* \widetilde{\pi}^* D$. We have the equality $\widetilde{\pi}^* D = \xi_{p_{i+1}} + E$ for some divisor E contracted by φ. Moreover, the equality $\widetilde{\pi}'_*(\xi_{p_{i+1}} + E) = D'$ holds. Therefore, D' is the strict transform of D. Similarly as before we get $D' \cdot \widetilde{\pi}_*[Y_{k_i'}] > 0$. That means D' is φ'-ample. □

Corollary 8.13. *(i) The varieties* $\widehat{X}(\widehat{w})$ *obtained from construction 3 are linked by flops.*

(ii) Any variety obtained from a construction 3 variety $\widehat{X}(\widehat{w})$ *via any flop is again of construction 3.*

Proof. The first part follows because the flops described in the last proposition link those varieties. All varieties from construction 3 are relative minimal models. Therefore, the only possibility to get a flop is the contraction of a curve $\widetilde{\pi}_*[Y_j]$ for $j \in m_{\widetilde{w}}(Q_w)$ with $K_{\widetilde{X}(\widetilde{w})} \cdot \widetilde{\pi}_*[Y_j] = 0$. We handled all those contractions in the previous proposition except for one. That is the map φ almost as above, but with the property $\beta_{p_i} = \alpha_{n-1}$ and $\beta_{p_{i+1}} = \alpha_n$. However, that contraction is not small because the divisor $D_{p_{i+1}}$ is contracted. □

Now we are able to prove the main result of this section.

Theorem 8.14. *The relative minimal models of $X_P(w)$ birational to $X_P(w)$ are exactly the varieties $\widetilde{X}(\widetilde{w})$ obtained from construction 3.*

Proof. We already showed that these varieties are relative minimal models birational to $X_P(w)$. The last two results show that the conjectures existence and termination of flops (see [Mat02]) hold for these minimal models. We want to use Theorem 12-1-8 in [Mat02] to get the result. That theorem requires existence and termination of flops to hold in general. However, everything used in the proof has already been proven by us in the situation of $X_P(w)$. □

8.2 The B_n-Case I

The next goal is to obtain the results of the last section in the B_n-case. We will do this separately for two sets of Schubert varieties. Let G be a semisimple linear algebraic group with root system B_n. By P we denote the maximal parabolic subgroup of G corresponding the simple root α_1 (see [Bou68] for notation). Let w be a cominuscule element of the Weyl group of G with $w \neq s_{\alpha_n} \cdots s_{\alpha_1}$. Let p be the unique peak of Q_w. We will show that $X_P(w)$ itself is the only relative minimal model of $X_P(w)$ birational to $X_P(w)$.

Proposition 8.15. *The variety $X_P(w)$ has only terminal singularities.*

Proof. By our assumption on w we get $\beta_p \neq \alpha_n$. By Proposition 6.19 we have the equality

$$-K_{X_P(w)} = \begin{cases} h(p)D_p & \text{, if } \exists t \in Q_w \text{ with } \beta_t = \alpha_n \text{ and } p \succ t \\ (h(p)+1)D_p & \text{, otherwise.} \end{cases}$$

As always $\pi : \widetilde{X}(\widetilde{w}) \to X_P(w)$ is the Bott-Samelson resolution. Because $\beta_p \neq \alpha_n$, the divisor D_p is the ample generator of the Picard group of $X_P(w)$ (for $\beta_p = \alpha_n$ it is $2D_P$). Therefore, we can use Proposition 6.15 to get the equality

$$\pi^*(D_p) = \sum_{i \in Q_w, \beta_i \neq \alpha_n} \xi_i + \sum_{i \in Q_w, \beta_i = \alpha_n} 2\xi_i.$$

Together with the description of $K_{\widetilde{X}(\widetilde{w})}$ in Corollary 6.16 we get

$$K_{\widetilde{X}(\widetilde{w})} - \pi^* K_{X_P(w)} = \begin{cases} 2(h(p)-h(t))\xi_t + \sum_{i\succ t}(h(p)-h(i))\xi_i \\ + \sum_{i\prec t}(h(p)-h(i)-1)\xi_i & \text{, if } \exists t \in Q_w \text{ with } \beta_t = \alpha_n \\ \sum_{i\in Q_w}(h(p)-h(i))\xi_i & \text{, otherwise.} \end{cases}$$

The coefficients of the ξ_i are all positive except for ξ_p which is not contracted by π. Therefore, $X_P(w)$ has only terminal singularities. □

We get the main result of this section.

Theorem 8.16. *The variety $X_P(w)$ is the unique relative minimal model of $X_P(w)$ which is birational to $X_P(w)$.*

Proof. The variety $X_P(w)$ is locally $\mathbb{Q}$-factorial (see Corollary 6.18) and has terminal singularities. Therefore, $X_P(w)$ is a relative minimal model of itself.

The identity obviously contracts no curves. Therefore, there are no flops. That means existence and termination of flops hold trivially for $X_P(w)$. As in the C_n-case we use Theorem 12-1-8 in [Mat02] to conclude the proof. □

8.3 The B_n-Case II

In this section we deal with the remaining cominuscule case. Let G be a semisimple linear algebraic group with root system B_n. By P we denote the maximal parabolic subgroup corresponding to the simple root α_1. Let w be the element of the Weyl group given by $s_{\alpha_n} \cdots s_{\alpha_1}$. Proposition 6.1 shows that w is cominuscule. In contrast to the rest of the B_n-case we will need to cut the quiver in order to get a relative minimal model. However, there will still be only a unique relative minimal model of $X_P(w)$ birational to $X_P(w)$. By $\widehat{w}$ we denote the cominuscule good reduced generalized decomposition given by $(s_{\alpha_n}, s_{\alpha_{n-1}} \cdots s_{\alpha_1})$. To keep notation of previous chapters we set $\beta_i = \alpha_{n-i+1}$ for all $i \in [1,n]$. Then $w = s_{\beta_1} \cdots s_{\beta_n}$ holds. With this notation the equalities $m_{\widehat{w}}(Q_w) = \{1, n\}$ and $p_{\widehat{w}}(Q_w) = \{1, 2\}$ hold.

Proposition 8.17. *We have the equalities*

$$\mathcal{L}_n = \sum_{i=2}^{n} \xi_i + 2\xi_1,\ \mathcal{L}_1 = \xi_1,$$
$$\mathcal{M}_n = \widehat{D}_2 + 2\widehat{D}_1,\ \mathcal{M}_1 = \widehat{D}_1.$$

Proof. The first two equalities are consequences of Proposition 4.36. The second two equalities are consequences of the first two. □

To compute the canonical divisor of $\widehat{X}(\widehat{w})$ we need the following proposition.

Proposition 8.18. *The morphism $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ is rational. In particular, there is an isomorphism $K_{\widehat{X}(\widehat{w})} \cong \widetilde{\pi}_* K_{\widetilde{X}(\widetilde{w})}$.*

Proof. As in Proposition 8.6. □

As a corollary we compute $K_{\widehat{X}(\widehat{w})}$.

Corollary 8.19. *We have the equalities*

$$-K_{\widehat{X}(\widehat{w})} = n\widehat{D}_2 + 2n\widehat{D}_1 = n\mathcal{M}_n.$$

In particular, the variety $\widehat{X}(\widehat{w})$ is Gorenstein.

Proof. This follows using the last proposition and Corollary 6.16. □

Proposition 8.20. *The variety $K_{\widehat{X}(\widehat{w})}$ has only terminal singularities.*

Proof. Using Corollary 6.16 we get

$$K_{\widetilde{X}(\widetilde{w})} - \widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i=2}^{n} (i-2)\xi_i.$$

The proposition follows because $i - 2 > 0$ for all $i \in [3, n]$. □

At this point we can prove the main theorem of this section.

Theorem 8.21. *The variety $\widehat{X}(\widehat{w})$ with $\widehat{w} = (s_{\alpha_n}, s_{\alpha_{n-1}} \cdots s_{\alpha_1})$ is the unique relative minimal model of $X_P(w)$ birational to $X_P(w)$.*

Proof. We already know that $\widehat{X}(\widehat{w})$ is locally factorial with only terminal singularities. Moreover, $K_{\widehat{X}(\widehat{w})}$ is $\widehat{\pi}$-nef because $K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_1] = 0$. Therefore, $\widehat{X}(\widehat{w})$ is indeed a relative minimal model of $X_P(w)$.

Contracting the curve $\widetilde{\pi}_*[Y_1]$ is done via $\widehat{\pi}$. But that morphism is not small because the divisor $\widehat{D}_2$ is contracted. Thus, this map doesn't give a flop. There are no other possibilities for a potential flop which means we have proven existence and termination of flops for $\widehat{X}(\widehat{w})$. As before we use Theorem 12-1-8 in [Mat02] to conclude the proof. □

9 IH-Small Resolutions of Cominuscule Schubert Varieties

In this chapter we classify all IH-small resolutions of cominuscule Schubert varieties in the B_n and C_n-case. The corresponding results in the minuscule case are to be found in [Per07]. The proofs work the same way in the cominuscule case.

Definition 9.1. A projective birational morphism of varieties $\pi : Y \to X$ is called IH-small (resp. IH-semismall) if for all $k > 0$ the following inequality holds.

$$\operatorname{codim}_X\{x \in X \mid \dim \pi^{-1}(x) = k\} > 2k \text{ (resp. } \geq 2k)$$

In addition, if Y is smooth, then π is called an IH-small (resp. IH-semismall) resolution.

There is the following result on IH-small resolutions due to Totaro.

Proposition 9.2 ([Tot00], Proposition 8.3)**.** *Let $\pi : Y \to X$ be an IH-small resolution. Then Y is a relative minimal model of X.*

As a first step towards classifying all IH-small resolution we need to check which relative minimal models are smooth. In order to do this we need to know which cominuscule Schubert varieties are smooth. The following result is due to [BP99].

Proposition 9.3. *A cominuscule Schubert variety $X_P(w)$ is smooth if and only if it is homogeneous under its stabilizer.*

Corollary 9.4. *A cominuscule Schubert variety $X_P(w)$ is smooth if and only if Q_w has no non virtual hole. In particular, a cominuscule Schubert variety $X_P(w)$ in the C_n-case is smooth if and only if $\beta(p(Q_w)) = \{\alpha_n\}$.*

Proof. This follows from the equality $\Sigma(P_w) = \beta(\mathrm{Holes}(Q_w))$. □

For any element $w \in W$ we denote the class of w in $X_{P^w}(w)$ as e_w. In the case of a Schubert variety we can reformulate the definition of IH-smallness as follows.

Lemma 9.5. *Let $\widehat{w}$ be a good reduced generalized decomposition of some element $w \in W$. Let P be the parabolic subgroup P^w. Then $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X_P(w)$ is IH-small (resp. IH-semismall) if and only if the following inequality holds for all $w' \leq w$ with $P_w \subseteq P_{w'}$ and $\dim \widehat{\pi}^{-1}(e_{w'}) \neq 0$.*

$$\operatorname{codim}_{X_P(w)} X_P(w') > 2 \dim \widehat{\pi}^{-1}(e_{w'}) \text{ (resp. } \geq)$$

Proof. Because $\widehat{w}$ is good the morphism $\widehat{\pi}$ is P_w-equivariant. Therefore, the set

$$\operatorname{codim}_{X_P(w)}\{x \in X \mid \dim \widehat{\pi}^{-1}(x) = k\}$$

is P_w-stable for all $k \geq 0$. Thus, the irreducible components of it's closure are Schubert varieties $X_P(v)$ with $P_w \subseteq P_v$. □

Let us recall the inductive definition of $\widehat{X}(\widehat{w})$ for some cominuscule good reduced generalized decomposition $\widehat{w} = (u, w_2, \ldots, w_{s+1})$. Let $v = w_2 \cdots w_{s+1}$. The first step can be given by the map $p : \overline{QuH} \times^H X_P(v) \to X_P(w)$, where $P = P^w$, $Q = P_w$, and $\Sigma(H) = \Sigma(P_v) \cup (\Sigma(P_w) \cap \operatorname{Supp}(u)^c)$. By induction there is a map $\pi' : \widehat{X}(\widehat{v}) \to X_P(v)$, which is P_v-equivariant. Putting this together gives the map $\widehat{\pi} : \overline{QuH} \times^H \widehat{X}(v) \to X_P(w)$.

Let $w' \leq w$ with $P_w \subseteq P_{w'}$. We denote the Q-orbit of $e_{w'}$ by $U(w')$. We want to calculate $f_{\widehat{\pi},w'} := \dim \widehat{\pi}^{-1}(e_{w'})$. Since $\widehat{\pi}$ is Q-equivariant, all the fibers of points in $U(w')$ are isomorphic and the equality $f_{\widehat{\pi},w'} = \dim \widehat{\pi}^{-1}(U(w')) - \dim U(w')$ holds. We define the set $S(w', w)$ by

$$S(w', w) = \{v' \in W/W_P \mid v' \leq v,\ QX_P(uv') = X_P(w'),\ HX_P(v') = X_P(v')\}.$$

Lemma 9.6. *Assume that $X_H(u)$ is smooth.*

(i) We have the equality

$$p^{-1}(U(w')) = \bigcup_{v' \in S(w',w)} QuH \times^H He_{v'}.$$

(ii) The equality

$$\widehat{\pi}^{-1}(U(w')) = \bigcup_{v' \in S(w',w)} QuH \times^H \pi'^{-1}(He_{v'})$$

holds.

(iii) In particular, there is an element $v' \in S(w', w)$ such that

$$\begin{aligned} f_{\widehat{\pi},w'} &= \#Q_u + f_{\pi',v'} + \#Q_{v'} - \#Q_{w'} \\ &= \#Q_u + f_{\pi',v'} - \operatorname{codim}_{X_P(w')}(X_P(v')). \end{aligned}$$

Proof. The second part follows directly from the first one, while the third statement is a direct consequence of the second one. Therefore, we are left to prove (i).

Let $v' \in S(w', w)$. Since $X_H(u)$ is smooth, the quiver Q_u has no holes. Thus, a representative of u is in Q. Moreover, the inclusion $H \subseteq Q$ holds. Therefore, we have the equality

$$p(QuH \times^H He_{v'}) = QuHe_{v'} = QuHu^{-1}e_{uv'} = U(w').$$

That implies the inclusion $QuH \times^H He_{v'} \subseteq p^{-1}(U(w'))$.

Let $(x, y) \in \overline{QuH} \times^H X_P(v)$ such that $p(x, y) \in U(w')$. We have the two equalities

$$\overline{QuH} = Q, \quad X_P(v) = \bigcup_{v' \leq v,\ HX_P(v') = X_P(v')} He_{v'}.$$

Therefore, $x \in Q$ and $y = he_{v'}$ for some $h \in H$ and $v' \leq v$ with $HX_P(v') = X_P(v')$. Since $p(x, y) \in U(w')$ there is an element $q \in Q$ such that $e_{w'} = qxhe_{v'} = qxhu^{-1}e_{uv'} \in Qe_{uv'}$. Therefore, we get the equality $QX_P(uv') = X_P(w')$. Thus, we have $v' \in S(w', w)$. □

9.1 The C_n-Case

The goal of this section is to prove the following theorem.

Theorem 9.7. *Let G be a semisimple linear algebraic group with root system C_n. By P we denote the maximal parabolic subgroup of G defined via $\Sigma(P) = \{\alpha_n\}$. Then the IH-small resolutions of a cominuscule Schubert variety $X_P(w)$ are exactly given by $\widehat{X}(\widehat{w}) \to X_P(w)$, where $\widehat{w}$ is obtained from construction 2 via an ordering $(p_1, \ldots, p_{s+1})$ of $p(Q_w)$ with $\beta_{p_{s+1}} = \alpha_n$.*

If the equality $p'(Q_w) \neq p(Q_w)$ holds, then there is no ordering of construction 2. Therefore, we have can assume $p(Q_w) = p'(Q_w)$ throughout the whole section. We already know that all IH-small resolutions are relative minimal models. Therefore, only resolutions $\widehat{X}(\widehat{w}) \to X_P(w)$ coming from construction 3 need to be checked. The first step will be to determine, which of those are smooth.

Proposition 9.8. *Let $\widehat{w}$ be a generalized decomposition obtained from construction 1. Then $\widehat{X}(\widehat{w})$ is smooth if and only if $\widehat{w}$ is obtained from an ordering $(p_1, \ldots, p_{s+1})$ of $p'(Q_w)$ such that $\beta_{p_{s+1}} = \alpha_n$.*

Proof. If $\beta_{p_{s+1}} \neq \alpha_n$, then Q_{w_s} has a peak different from α_n. Moreover, we have $\beta_m = \alpha_n$, where m is the minimal vertex in $Q_{w_{s+1}}$. By Corollary 9.4 the variety $X_P(w_{s+1})$ is not smooth. Thus, $\widehat{X}(\widehat{w})$ is not smooth.

If we have the equality $\beta_{s+1} = \alpha_n$, then $Q_{w_{s+1}}$ has no non virtual hole. Indeed, the only non virtual hole could be a vertex $j \in Q_{w_{s+1}}$ with $\beta_j = \alpha_n$. Therefore, the variety $X_P(w_{s+1})$ is smooth. All the other quivers Q_{w_i} for $i \in [1, s]$ give Schubert varieties in type A_n. Any quiver of a cominuscule Schubert variety in the A_n-case with a unique peak has no non-virtual holes. The lemma follows. □

The next step is to analyze the set $S(w', w)$ as defined in the previous chapter for $w' \leq w$ with $P_w \subseteq P_{w'}$.

Remark 9.9. (i) The condition $HX_P(v') = X_P(v')$ is equivalent to $H \subseteq P_{v'}$. That is equivalent to $\beta(\text{Holes}(Q_{v'})) \subseteq \beta(\text{Holes}(Q_v))$.

(ii) Let i be a hole of $Q_{w'}$ with the property $\beta_i \notin \mathrm{supp}(u)$. That means no representative of s_{β_i} is in Q. Therefore, the equality $QX_P(uv') = X_P(w')$ implies $i \in Q_{v'}$.

We will use the following strategy to prove that the resolutions $\widehat{X}(\widehat{w}) \to X_P(w)$, where $\widehat{w}$ is obtained from construction 2 via an ordering $(p_1, \ldots, p_{s+1})$ of $p(Q_w)$ with $\beta_{p_{s+1}} = \alpha_n$, are small. The goal is to show the inequality $\mathrm{codim}_{X_P(w)}(X_P(w')) > 2f_{\widehat{\pi},w'}$ for all $w' \leq w$ such that $X_P(w')$ is stable under P_w. We will introduce two functions Γ and q such that $\mathrm{codim}_{X_P(w)}(X_P(w')) = \Gamma(w', w) + q(w', w)$. Moreover, the function q fulfills $q(w', w) \geq 0$ with equality if and only if $w = w'$. We will prove the stronger statement $\Gamma(w', w) \geq 2f_{\widehat{\pi},w'}$ by induction on $\#p(Q_w)$. Using Lemma 9.6, we get the equality

$$f_{\widehat{\pi},w'} = \#Q_u + f_{\pi',v'} - \mathrm{codim}_{X_P(w')}(X_P(v')).$$

Let $\theta \in W$ be defined by $P_v X_P(v') = X_P(\theta)$. Because π' is P_v-stable, we have the equality $f_{\pi',v'} = f_{\pi',\theta}$. By induction the inequality $\Gamma(\theta, v) \geq 2f_{\pi',\theta}$ holds. Therefore, we will be done by proving the following inequality.

$$\Gamma(w', w) - \Gamma(\theta, v) \geq 2(\#Q_u - \mathrm{codim}_{X_P(w')}(X_P(v'))) \tag{9.1}$$

The functions Γ and q

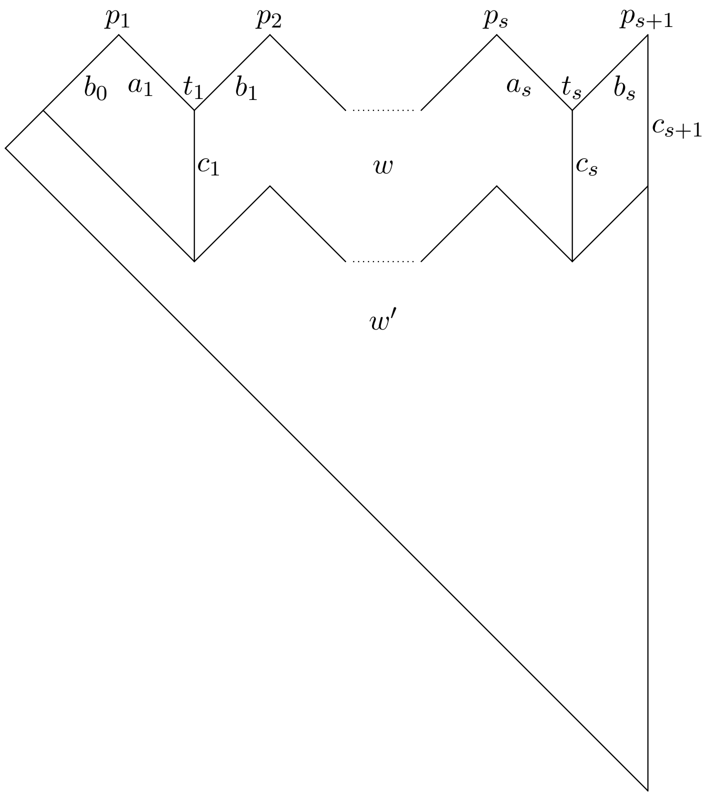
Let $p_1, \ldots, p_{s+1}$ be the peaks of Q_w and $\alpha_{k_i} = \beta_{p_i}$ for $i \in [1, s+1]$. Without loss of generality we may assume $k_1 < \ldots < k_{s+1}$. Moreover, let t_i be the hole between p_i and p_{i+1} for all $i \in [1, s]$. We define the following sequences of integers.

$$a_i(w) := h(p_i) - h(t_i), \ i \in [1, s]$$
$$b_i(w) := h(p_{i+1}) - h(t_i), \ i \in [1, s]$$
$$b_0(w) := \frac{1}{2}(h(p_{s+1}) - 1) - \sum_{i=1}^{s} b_i(w)$$

For any $i \in [1, s]$ we define the depth of the hole t_i in w' as

$$c_i := \#\{j \in Q_w \backslash Q_{w'} \mid \beta_j = \beta_{t_i}\}.$$

Additionally, we set $c_0 = 0$ and $c_{s+1} = c_s$.



The sequences a_i and b_i for w' are given as follows.

$$a_i(w') = a_i(w) + c_i - c_{i-1}, \ \ i \in [1, s]$$
$$b_i(w') = b_i(w) + c_i - c_{i+1}, \ \ i \in [0, s]$$

Since $Q_{w'}$ can have less holes than Q_w, we slightly abuse notation here. However, in that case we have just some additional zeros. The functions Γ and q are given by

$$\Gamma(w', w) = \sum_{i=1}^{s} c_i(a_i + b_i) + \frac{1}{2} c_s,$$
$$q(w', w) = \frac{1}{2} \sum_{i=1}^{s} (c_i - c_{i-1})^2.$$

Lemma 9.10. *The equality* $\operatorname{codim}_{X_P(w)}(X_P(w')) = \Gamma(w', w) + q(w', w)$ *holds. Moreover, we have* $q(w', w) \geq 0$ *with equality if and only if* $w = w'$.

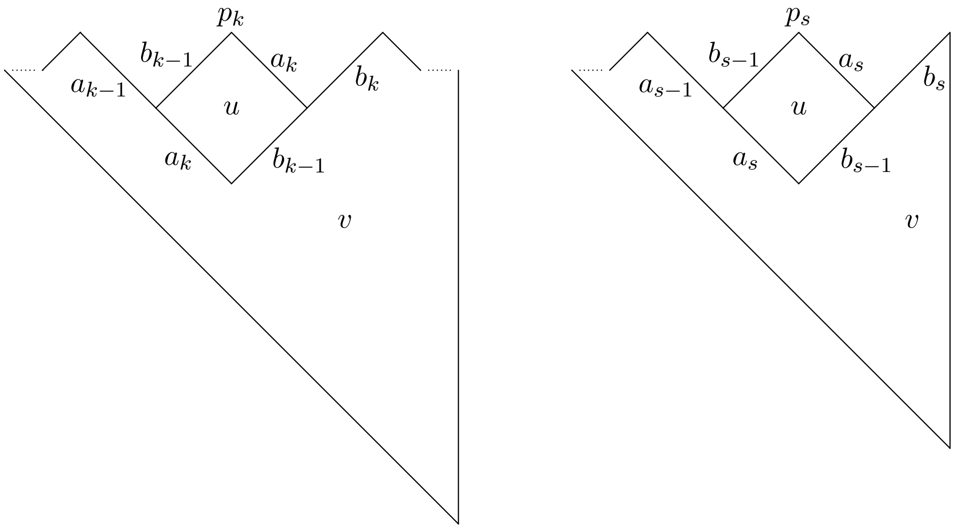
Proof. Without much difficulty we get the equality

$$\operatorname{codim}_{X_P(w)}(X_P(w')) = \sum_{i=1}^{s} c_i(a_i + b_i) + \sum_{i=1}^{s} c_i^2 - \sum_{i=1}^{s-1} c_i c_{i+1} - \frac{1}{2} c_s(c_s - 1).$$

The lemma follows. □

Proof of Theorem 9.7

Let p_k be the peak of Q_u for some $k \in [1, s+1]$. We need to deal with two cases. One case is given by $k < s$ and the other case is $k = s$. We proof both cases simultaneously because they don't differ very much.

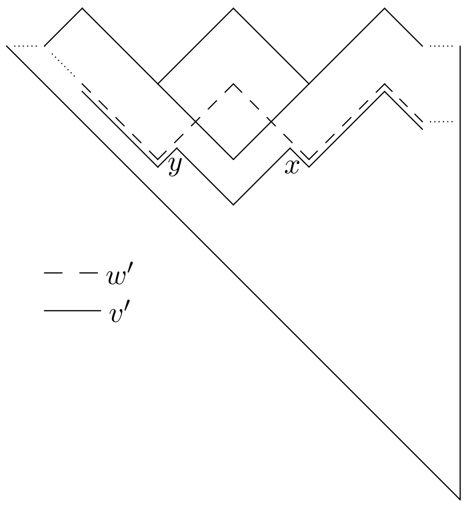


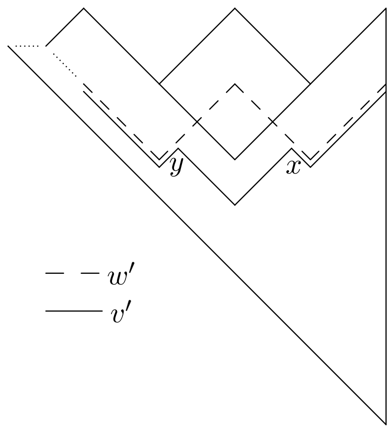
The quiver Q_v has one peak and one hole less than Q_w. The values $a_i(v)$ and $b_i(v)$ are given as follows.

$$a_i(v) = \begin{cases} a_i(w) & \text{, if } i \in [1, k-2] \\ a_k(w) + a_{k-1}(w) & \text{, if } i = k-1 \text{ and } i > 0 \\ a_{i+1}(w) & \text{, if } i \in [k, s-1] \end{cases}$$

$$b_i(v) = \begin{cases} b_i(w) & \text{, if } i \in [0, k-2] \\ b_k(w) + b_{k-1}(w) & \text{, if } i = k-1 \\ b_{i+1}(w) & \text{, if } i \in [k, s-1] \end{cases}$$

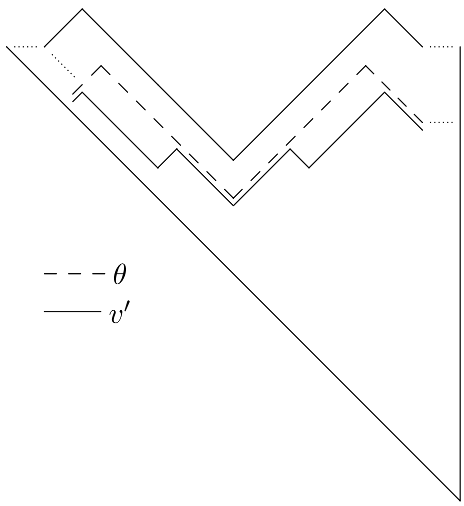
Recall that any hole i of w' with $\beta_i \notin \mathrm{Supp}(u)$ is in v'. Moreover, v' may have an additional hole corresponding to the hole of v that is not in w. We define $x := a_{k+1}(v')$ and $y := b_{k-1}(v')$.

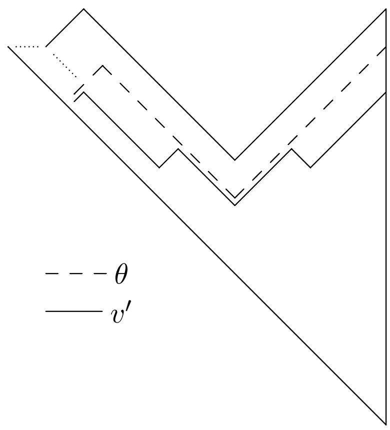




Because $v' \leq v$, we have $x \in [0, c_k]$ and $y \in [0, c_{k-1}]$. Due to the fact that the only hole of $Q_{v'}$ which is not a hole of $Q_{w'}$ corresponds to the same root as the minimal element of u, we get the equality $c_{k-1} - y = c_k - x$.

In order to prove the inequality (9.1) we need to describe the values a_i, b_i and c_i for θ. Recall that θ was defined by $P_v X_P(v') = X_P(\theta)$.





We define the depth c_i' of θ in v in the same way as the depth c_i of w' in w. We have the equality

$$c_i' = \begin{cases} c_i & , \ i \in [1, k-2] \\ c_{k-1} - y = c_k - x & , \ i = k-1 \\ c_{i+1} & , \ i \in [k, s-1]. \end{cases}$$

Moreover, the equality $c_s' = c_{s-1}'$ holds. In the case $k < s$ this means $c_s' = c_{s+1}$, while in the case $k = s$ we have $c'_s = c_s - y = c_{s+1} - x$. Without much difficulty, we get the equalities

$$\begin{aligned}\#Q_u - \operatorname{codim}_{X_P(w')}(X_P(v')) &= a_k(w)b_{k-1}(w) - (b_{k-1}(w') - y)(a_k(w') - x)\\ &= a_k(w)b_{k-1}(w) - (b_{k-1}(w) - x)(a_k(w) - y)\\ &= xa_k(w) + yb_{k-1}(w) - xy.\end{aligned}$$

The right hand side of equation (9.1) is given by

$$\begin{aligned}\Gamma(w', w) - \Gamma(\theta, v) &= \sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) + \frac{1}{2}c_s - \sum_{i=1}^{s-1} c'_i(a_i(v) + b_i(v)) - \frac{1}{2}c'_{s-1}\\ &= \sum_{i=k-1}^{k} c_i(a_i(w) + b_i(w)) - c'_{k-1}(a_{k-1}(v) + b_{k-1}(v)) + \frac{1}{2}c_s - \frac{1}{2}c'_{s-1}\\ &= x(a_k(w) + b_k(w)) + y(a_{k-1}(w) + b_{k-1}(w)) + \frac{1}{2}c_s - \frac{1}{2}c'_{s-1}.\end{aligned}$$

The fact that $\widehat{w}$ is obtained by construction 2 implies the inequalities $a_k(w) \leq b_k(w)$ and $b_{k-1}(w) \leq a_{k-1}(w)$. Moreover, we have

$$\frac{1}{2}(c_s - c'_{s-1}) = \begin{cases} 0 & \text{, if } k < s\\ \frac{1}{2}x & \text{, if } k = s.\end{cases}$$

We can conclude one direction of the proof with the inequality

$$x(a_k(w) + b_k(w)) + y(a_{k-1}(w) + b_{k-1}(w)) + \frac{1}{2}(c_s - c'_{s-1}) \geq 2(xa_k(w) + yb_{k-1}(w) - xy).$$

We are left to prove, that the other smooth minimal models are not IH-small. This is a special case of the following lemma.

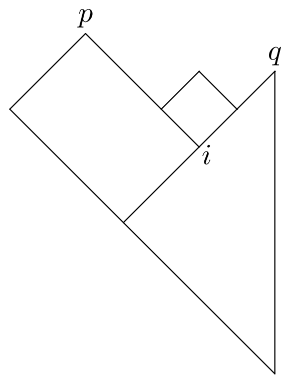
Lemma 9.11. *If $\widehat{w}$ is of construction 1 but not of construction 2, then $\widehat{X}(\widehat{w}) \to X_P(w)$ is not small.*

Proof. We proof the lemma in two cases.

(i) Assume $\alpha_{n-1} \in p(Q_w)$. Let $q \in p'(Q_w)$ with $\beta_q = \alpha_n$. Then the image $\widetilde{\pi}(Z_q)$ is a divisor in $\widehat{X}(\widehat{w})$. The image $\pi(Z_q)$ is some Schubert variety $X_P(w')$ of codimension 2 in $X_P(w)$. The fact that $\widehat{\pi}^{-1}(X_P(w'))$ contains $\widetilde{\pi}(Z_q)$ implies the inequality $f_{\widehat{\pi},w'} \geq 1$. We can conclude in this case via

$$\operatorname{codim}_{X_P(w)}(X_P(w')) = 2 \leq 2f_{\widehat{\pi},w'}.$$

(ii) In the second case we assume $\alpha_{n-1} \notin p(Q_w)$. Since $\widehat{w}$ is not of construction 2, we can find peaks $p, q \in p(Q_w)$ such that $q = f(p)$ and $h(q) < h(p)$. Let $i \in Q_w$ be the maximal vertex smaller than p and q.



The inequality $h(q)-h(i) < h(p)-h(i)$ holds. Moreover, the divisor $\widetilde{\pi}(Z_i)$ is of codimension $h(q)-h(i)+1$ in $\widehat{X}(\widehat{w})$. The image $\pi(Z_i)$ is a Schubert variety $X_P(w')$ of codimension $h(q) - h(i) + h(p) - h(i) + 1$ in $X_P(w)$. Since $\widehat{\pi}^{-1}(X_P(w'))$ contains $\widehat{\pi}(Z_i)$ we get the inequality $f_{\widehat{\pi},w'} \geq h(p) - h(i)$. We can conclude the proof because the inequality

$$\begin{aligned} \operatorname{codim}_{X_P(w)}(X_P(w')) &= h(q) - h(i) + h(p) - h(i) + 1 \\ &\leq 2(h(p) - h(i)) \\ &\leq 2 f_{\widehat{\pi},w'} \end{aligned}$$

holds. □

9.2 The B_n-Case

Due to the results of the sections 8.2 and 8.3 the B_n-case is way easier than the C_n-case. We get the following theorem.

Theorem 9.12. *Let G be a semisimple linear algebraic group with root system B_n. By P we denote the maximal parabolic subgroup of G defined via $\Sigma(P) = \{\alpha_1\}$. A cominuscule Schubert variety $X_P(w)$ admits a small resolution if and only if it is already smooth itself. Moreover, in that case the only small resolution is given by the identity $X_P(w) \to X_P(w)$.*

Proof. Obviously, the identity is always a small map. That already proves one direction. Assume that $X_P(w)$ is not smooth. We need to check that no minimal model is smooth and yields a small resolution at the same time.

Assume $w \neq s_{\alpha_n} \cdots s_{\alpha_1}$. By Theorem 8.16 the variety $X_P(w)$ is the only relative minimal of model $X_P(w)$ birational to $X_P(w)$. Therefore, we can conclude the proof in this case by Proposition 9.2.

We are left to deal with the case $w = s_{\alpha_n} \cdots s_{\alpha_1}$. By Theorem 8.21 the variety $\widehat{X}(\widehat{w})$ with $\widehat{w} = (s_{\alpha_n}, s_{\alpha_{n-1}} \cdots s_{\alpha_1})$ is the unique relative minimal model of $X_P(w)$ birational to $X_P(w)$. Let $w' = v' = s_{\alpha_{n-2}} \cdots s_{\alpha_1}$. Using Lemma 9.6 yields the equality

$$f_{\widehat{\pi},w'} = \#Q_u + f_{\pi',v'} - \operatorname{codim}_{X_P(w')}(X_P(v')) = 1.$$

Therefore, the equality

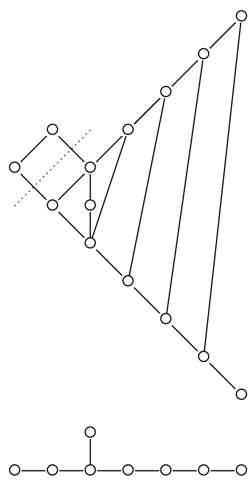
$$\operatorname{codim}_{X_P(w)}(X_P(w')) = 2 = 2f_{\widehat{\pi},w'}$$

shows that $\widehat{X}(\widehat{w}) \to X_P(w)$ is not small. □

10 Further Ideas

We are going to examine several directions in which the results might be generalized. Assume that G is any Kac-Moody group and let Λ be a dominant weight. Moreover, we define the parabolic subgroup $P \subseteq G$ by $\Sigma(P) = \{\alpha \in S \mid \langle \alpha^\vee, \Lambda \rangle \neq 0\}$. We say that an element w of W is Λ-minuscule if there exists a reduced decomposition $w = s_{\beta_1} \cdots s_{\beta_r}$ such that $\langle \beta_{i-1}^\vee, s_{\beta_i} \cdots s_{\beta_r}(\Lambda) \rangle = -1$ for all $i \in [2, r+1]$ (compare with Lemma 5.4). A natural step would be to classify small resolutions of Λ-minuscule Schubert varieties. There are even Λ-minuscule Schubert varieties over semisimple linear algebraic groups that are not minuscule.

Example 10.1. The following quivers corresponds to a Λ-minuscule Schubert variety in the E_8-case. This cut of the quiver yields a small resolution.



The case by case study used by Perrin in [Per07] to show smallness is however not suitable here. There are far too much cases to check.

Similar to the definition of cominuscule elements we can define Λ-cominuscule elements and try to extend the results of this thesis to the class of Λ-cominuscule Schubert varieties. In this thesis there was more case by case study used as in [Per07]. Thus, there is an even bigger need for more general arguments. Having small resolutions of Λ-(co)minuscule Schubert varieties it is likely that one can mimic the calculation of Kazdhan-Lusztig polynomials in [Zel83] and [SV95].

Another natural goal is to extend the results to non-(co)minuscule Schubert varieties.

There occur several problems. An advantage in the (co)minuscule case was the fact that elements had an unique reduced expression up to commuting relations. Indeed, we will get several quivers in the general case. To get a complete list of small resolutions one is probably forced to look at all the quivers. Moreover, we had a very nice description of the quivers due to Lemma 5.5. Getting a general description of the quivers is not an easy problem. Understanding the Bruhat order is also more complicated in the general case.

A completely different direction could be to generalize the result to other fields. While we computed small resolutions for any algebraically closed field, the complete classification was only proven over $\mathbb{C}$. One of the main issues in making Mori theory viable in characteristic p is the lack of cohomology vanishing results and the fact that resolution of singularities has not been proven in positive characteristic. However, Schubert varieties can always be desingularized by the Bott-Samelson resolution. The fact, that any Schubert variety is Frobenius split (see [BK05]) gives many vanishing results. Not only from the perspective of this thesis, but also from the general desire for a Mori theory in characteristic p, it might be worthwhile to investigate, whether parts of the program work under the assumptions of Frobenius splitting and resolution of singularities.

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