

COUNTEREXAMPLE TO THE GENERALIZED BOGOMOLOV-GIESEKER INEQUALITY FOR THREEFOLDS

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ABSTRACT. We give a counterexample to the generalized Bogomolov-Gieseker inequality for threefolds conjectured by Bayer, Macrì and Toda using the blow up of a point in three dimensional projective space.

1. INTRODUCTION

Tom Bridgeland introduced the notion of a stability condition on a triangulated category in [Bri07]. He was inspired by the study of Dirichlet branes in string theory by Douglas in [Dou02]. Instead of defining stability such as slope stability or Gieseker stability in the category of coherent sheaves, one uses other abelian categories inside the bounded derived category of coherent sheaves. One of the main problems is the construction of stability conditions on smooth projective threefolds.

For surfaces the construction was carried out in [Bri08] and [AB13]. The key ingredient is the classical Bogomolov-Gieseker inequality for semistable sheaves. In [BMT14] Bayer, Macrì and Toda proposed a conjectural construction for threefolds based on a generalized Bogomolov-Gieseker inequality involving the third Chern character. It was proved for \mathbb{P}^3 in [Mac14b], for the smooth quadric hypersurface Q in \mathbb{P}^4 in [Sch14] and for abelian threefolds independently in [MP16] and [BMS14]. Moreover, the cases of \mathbb{P}^3 and Q were recently generalized to all Fano threefolds of Picard rank one in [Li15].

Let X be the blow up of \mathbb{P}^3 in one point. The article [Li15] contained the conjecture for X as a further example. Unfortunately that example contained a mistake, which lead us directly to the counterexample. In order to do so, we choose the polarization to be $H = -\frac{1}{2}K_X$ and the object that does not satisfy the conjecture is given by the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$.

Acknowledgements. The author would like to thank Arend Bayer, Chunyi Li, Emanuele Macrì, Yukinobu Toda, Bingyu Xia, Xiaolei Zhao and the referee for commenting on previous versions of this article. The author is partially supported by NSF grant DMS-1523496 (PI Emanuele Macrì) and a Presidential Fellowship of the Ohio State University.

2. PRELIMINARIES

We start by recalling the notion of tilt stability due to [BMT14]. Let X be a smooth projective threefold over the complex numbers and let H be an ample divisor on X . For $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}$ and $E \in D^b(X)$ we define

$$\nu_{\alpha,\beta}(E) = \frac{H \cdot \text{ch}_2^\beta(E) - \frac{\alpha^2}{2} H^3 \cdot \text{ch}_0^\beta(E)}{H^2 \cdot \text{ch}_1^\beta(E)},$$

2010 *Mathematics Subject Classification.* 14F05 (Primary); 14J30, 18E30 (Secondary).

Key words and phrases. Stability conditions, Derived categories, Threefolds.

where $\text{ch}^\beta(E) = e^{-\beta H} \cdot \text{ch}(E)$. A torsion pair is defined by

$$\begin{aligned}\mathcal{T}_\beta &= \{E \in \text{Coh}(X) : \forall E \rightarrow G : H^2 \cdot \text{ch}_1(G) > \beta H^3 \cdot \text{ch}_0(G) \text{ or } H^2 \cdot \text{ch}_1(G) = \text{ch}_0(G) = 0\}, \\ \mathcal{F}_\beta &= \{E \in \text{Coh}(X) : \forall F \subset E : H^2 \cdot \text{ch}_1(G) \leq \beta H^3 \cdot \text{ch}_0(G) \text{ and } \text{ch}_0(G) \neq 0\}.\end{aligned}$$

A new heart of a bounded t-structure is then defined as the extension closure $\text{Coh}^\beta(X) = \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$.

Definition 2.1. An object $E \in \text{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$ -semistable if for all subobjects $F \hookrightarrow E$ in $\text{Coh}^\beta(X)$ the inequality $\nu_{\alpha,\beta}(F) \leq \nu_{\alpha,\beta}(E)$ holds.

Bayer, Macrì and Toda conjectured an inequality for tilt semistable objects involving all three Chern characters in [BMT14]. Over time their conjecture has evolved into the following form. We define

$$\begin{aligned}\Delta(E) &= (H^2 \cdot \text{ch}_1(E))^2 - 2(H^3 \cdot \text{ch}_0(E))(H \cdot \text{ch}_2(E)), \\ Q_{\alpha,\beta}(E) &= \alpha^2 \Delta(E) + 4(H \cdot \text{ch}_2^\beta(E))^2 - 6(H^2 \cdot \text{ch}_1^\beta(E)) \text{ch}_3^\beta(E).\end{aligned}$$

Conjecture 2.2 ([BMT14, BMS14]). *Any $\nu_{\alpha,\beta}$ -semistable object $E \in \text{Coh}^\beta(X)$ satisfies $Q_{\alpha,\beta}(E) \geq 0$.*

In order to prove the counterexample in the next section, we need to have a better understanding of the structure of walls in tilt stability. A *numerical wall* is a non trivial zero set of a non trivial equation of the form $\nu_{\alpha,\beta}(v) = \nu_{\alpha,\beta}(w)$ for fixed $v, w \in K_0(X)$. A subset of a numerical wall is called an *actual wall* with respect to $v \in K_0(X)$ if the set of $\nu_{\alpha,\beta}$ -semistable objects with class v changes at it. Walls in tilt stability satisfy Bertram's Nested Wall Theorem. For surfaces it was proven in [Mac14a]. It still holds in the threefolds case as for example shown in [Sch15].

Theorem 2.3 (Structure Theorem for Walls in Tilt Stability). *Let $v \in K_0(X)$ be fixed. All numerical walls in the following statements are with respect to v .*

- (1) *Numerical walls in tilt stability are of the form*

$$x\alpha^2 + x\beta^2 + y\beta + z = 0,$$

for $x, y, z \in \mathbb{R}$. In particular, they are either semicircles with center on the β -axis or vertical rays.

- (2) *Numerical walls only intersect if they are identical.*
(3) *If a numerical wall has a single point at which it is an actual wall, then all of it is an actual wall.*
(4) *The equation $Q_{\alpha,\beta}(v) = 0$ is a numerical wall for v .*

3. COUNTEREXAMPLE

Let $f : X \rightarrow \mathbb{P}^3$ be the blow up of \mathbb{P}^3 in a point P . The Picard group of X is well known to be a free abelian group with two generators $\mathcal{O}(L) = f^* \mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}(E)$, where $E = f^{-1}(P)$. The variety X is Fano with canonical divisor given by $-4L + 2E$. In particular, $H = 2L - E$ is an ample divisor. We also have the intersection products $L^3 = E^3 = 1$, $L \cdot E = 0$ and $H^3 = 7$. The goal of this section is to prove the following counterexample to the conjectural inequality.

Theorem 3.1. *There exists $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that the line bundle $\mathcal{O}_X(L)$ is $\nu_{\alpha,\beta}$ -semistable, but $Q_{\alpha,\beta}(\mathcal{O}_X(L)) < 0$.*

Proof. Since $\mathcal{O}(L)$ is a line bundle it is a slope stable sheaf. In particular, either $\mathcal{O}(L)$ or $\mathcal{O}(L)[1]$ is $\nu_{\alpha,\beta}$ -stable for all $\alpha \gg 0$.

A straightforward computation shows $H^3 \cdot \text{ch}_0(\mathcal{O}(L)) = 7$, $H^2 \cdot \text{ch}_1(\mathcal{O}(L)) = 4$, $H \cdot \text{ch}_2(\mathcal{O}(L)) = 1$ and $\text{ch}_3(\mathcal{O}(L)) = \frac{1}{6}$. In particular, this means $H^2 \cdot \text{ch}_1^{1/2}(\mathcal{O}(L)) = \frac{1}{2}$. If $F \hookrightarrow \mathcal{O}(L)$ destabilizes

along the line $\beta = \frac{1}{2}$, we must have $\text{ch}_1^{1/2}(F) \in \{0, \frac{1}{2}\}$. That means either F or the quotient has slope infinity independently of α , a contradiction.

A completely numerical computation shows that $Q_{\alpha,\beta}(\mathcal{O}(L)) \geq 0$ is equivalent to the inequality

$$\alpha^2 + \left(\beta - \frac{1}{4}\right)^2 \geq \frac{1}{16}.$$

We are done if we can prove that there is no wall with equality in this inequality for $\mathcal{O}(L)$. Assume there is a destabilizing sequence $0 \rightarrow F \rightarrow \mathcal{O}(L) \rightarrow G \rightarrow 0$ giving exactly this wall. Taking the long exact sequence in cohomology, we get $H^3 \cdot \text{ch}_0(F) \geq 7$. By definition of $\text{Coh}^\beta(X)$ we have the inequalities $H^2 \cdot \text{ch}_1^\beta(\mathcal{O}(L)) \geq H^2 \cdot \text{ch}_1^\beta(F) \geq 0$ for all $\beta \in [0, \frac{1}{2}]$. This can be rewritten as

$$4 + \beta(H^3 \cdot \text{ch}_0(F) - 7) \geq H^2 \cdot \text{ch}_1(F) \geq \beta H^3 \cdot \text{ch}_0(F).$$

Notice that the middle term is independent of β and we can vary β independently on the left and right. Therefore, we get $4 \geq H^2 \cdot \text{ch}_1(F) \geq H^3 \cdot \text{ch}_0(F)/2$. This means $H^2 \cdot \text{ch}_1(F) = 4$ and $H^3 \cdot \text{ch}_0(F) = 7$. This does not give the correct wall. \square

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