

CONSTRUCTING STABILITY CONDITIONS ON THREEFOLDS

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ABSTRACT. Arend Bayer, Emanuele Macrì and Yukinobu Toda conjectured a method for constructing Bridgeland stability conditions on smooth projective threefolds over the complex numbers. They reduced the problem to a generalized Bogomolov-Gieseker type inequality involving the third Chern character. It has been proven in several special cases, but I recently found a counterexample.

I will recall the conjectural construction and present the counterexample. Then I will give a proof for \mathbb{P}^3 that comes from a more general proof for the case of Fano threefolds of Picard rank 1 due to Chunyi Li. These are notes of a talk given in Yau's student seminar at Harvard University in March 2016.

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1. SLOPE STABILITY

Let X be a smooth projective threefold over the complex numbers and H be an ample divisor on X . The notion of slope stability was introduced in order to study moduli spaces of sheaves.

For any $E \in \text{Coh}(X)$, we define the *slope* of E to be

$$\mu_H(E) = \frac{H^2 \text{ch}_1(E)}{H^3 \text{ch}_0(E)},$$

where dividing by 0 is interpreted as $+\infty$. An object $E \in \text{Coh}(X)$ is called *(semi)stable* if for all proper non trivial subsheaves $F \hookrightarrow E$ the inequality

$$\mu(F) < (\leq) \mu(E/F)$$

holds.

The motivation for this definition is that if one fixes a class $v \in K_0(X)$ such that all semistable E with Chern character v are stable, it is well known that there is a coarse moduli spaces parametrizing them (see for example [HL10][Theorem 4.3.4]).

The notion of slope stability is somewhat limited in several ways. Slope stability only makes use of the rank and the first Chern character of a sheaf. Even more refined notions such as Gieseker stability only use higher Chern character lightly. This makes it difficult to pick up geometry in high codimension with it. Additionally it is often times desirable to vary the polarization H to obtain different related moduli spaces. This is impossible if the Picard rank is one, making the notion too rigid.

Bridgeland's idea from [Bri07] is to vary the category $\text{Coh}(X)$ within the bounded derived category $D^b(X)$ and not only the slope function. Recall that the objects in $D^b(X)$ are bounded complex of coherent sheaves on X . The morphisms are morphisms of complexes up to quasi-isomorphisms.

2. TILT STABILITY

In order to simplify notation we denote the twisted Chern character $\text{ch}^\beta = e^{-\beta H} \cdot \text{ch}$ for any $\beta \in \mathbb{R}$. More detailed, one gets

$$\begin{aligned}\text{ch}_0^\beta &= \text{ch}_0, \\ \text{ch}_1^\beta &= \text{ch}_1 - \beta H \cdot \text{ch}_0, \\ \text{ch}_2^\beta &= \text{ch}_2 - \beta H \cdot \text{ch}_1 + \frac{\beta^2}{2} H^2 \cdot \text{ch}_0, \\ \text{ch}_3^\beta &= \text{ch}_3 - \beta H \cdot \text{ch}_2 + \frac{\beta^2}{2} H^2 \cdot \text{ch}_1 - \frac{\beta^3}{6} H^3 \cdot \text{ch}_0.\end{aligned}$$

The process of tilting is used to obtain a new abelian category. For more information on the general theory of tilting we refer to [HRS96]. A torsion pair is defined by

$$\begin{aligned}\mathcal{T}_\beta &= \{E \in \text{Coh}(X) : \text{any quotient } E \rightarrow G \text{ satisfies } \mu(G) > \beta\}, \\ \mathcal{F}_\beta &= \{E \in \text{Coh}(X) : \text{any subsheaf } F \subset E \text{ satisfies } \mu(F) \leq \beta\}.\end{aligned}$$

A new category is defined as the extension closure $\text{Coh}^\beta(X) := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$. It turns out to be abelian by general abstract theory. Let $\alpha > 0$ be a positive real number. The new slope function is

$$\nu_{\alpha,\beta} = \frac{H \cdot \text{ch}_2^\beta - \frac{\alpha^2}{2} H^3 \cdot \text{ch}_0^\beta}{H^2 \cdot \text{ch}_1^\beta}.$$

As before, an object $E \in \text{Coh}^\beta(X)$ is called *tilt-(semi)stable* if for all proper non trivial subobjects $F \hookrightarrow E$ the inequality

$$\nu_{\alpha,\beta}(F) < (\leq) \nu_{\alpha,\beta}(E/F)$$

holds. Without even varying the polarization H this provides already a two dimensional space for varying stability. However, we are still not using all Chern characters.

3. THE CONJECTURE AND BRIDGELAND STABILITY

Bayer, Macrì and Toda proposed that repeating the previous process with tilt stability instead of slope stability could allow to construct a so called *Bridgeland stability condition*. Let

$$\begin{aligned}\mathcal{T}_{\alpha,\beta} &= \{E \in \text{Coh}^\beta(X) : \text{any quotient } E \rightarrow G \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\}, \\ \mathcal{F}_{\alpha,\beta} &= \{E \in \text{Coh}^\beta(X) : \text{any subobject } F \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta}(F) \leq 0\}\end{aligned}$$

and set $\mathcal{A}^{\alpha,\beta}(X) = \langle \mathcal{F}_{\alpha,\beta}[1], \mathcal{T}_{\alpha,\beta} \rangle$. For any $s > 0$ they define

$$\lambda_{\alpha,\beta,s} = \frac{\text{ch}_3^\beta - (s + \frac{1}{6})\alpha^2 H^2 \cdot \text{ch}_1^\beta}{H \cdot \text{ch}_2^\beta - \frac{\alpha^2}{2} H^3 \cdot \text{ch}_0^\beta}.$$

Theorem 3.1 ([BMT14]). *The pair $(\mathcal{A}^{\alpha,\beta}(X), \lambda_{\alpha,\beta,s})$ is Bridgeland stability condition for all $s > 0$ if and only if for all tilt stable objects $E \in \text{Coh}^\beta(X)$ the inequality*

$$Q_{\alpha,\beta}(E) = \alpha^2((H^2 \text{ch}_1^\beta(E))^2 - 2(H^3 \text{ch}_0^\beta(E))(H \text{ch}_2^\beta(E))) + 4(H \text{ch}_2^\beta(E))^2 - 6(H^2 \text{ch}_1^\beta(E)) \text{ch}_3^\beta(E) \geq 0$$

holds

They conjectured that this inequality is true for all smooth projective threefolds. This turned out to be true in various cases. The first proof was in the case of \mathbb{P}^3 by [Mac14b] and a very similar proof worked for the smooth quadric hypersurface in \mathbb{P}^4 in [Sch14]. These results were generalized with a fundamentally different proof to all Fano threefolds of Picard rank one in [Li15]. The conjecture is also known to be true for all abelian threefolds with two independent proofs by [MP16] and [BMS14].

4. COUNTEREXAMPLE

Let $f : X \rightarrow \mathbb{P}^3$ be the blow up of \mathbb{P}^3 in a point P . The Picard group of X is well known to be a free abelian group with two generators $\mathcal{O}(L) = f^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}(E)$, where $E = f^{-1}(P)$. The variety X is Fano with canonical divisor given by $-4L + 2E$. In particular, $H = 2L - E$ is an ample divisor. We also have the intersection products $L^3 = E^3 = 1$, $L \cdot E = 0$ and $H^3 = 7$.

Before we can give the counterexample, we need to recall the structure of walls in tilt stability. For surfaces it was proven in [Mac14a]. It still holds in the threefolds case as for example shown in [Sch15].

Theorem 4.1 (Structure Theorem for Walls in Tilt Stability). *Let $v \in K_0(X)$ be fixed. All numerical walls in the following statements are with respect to v .*

- (1) *Numerical walls in tilt stability are of the form*

$$x\alpha^2 + x\beta^2 + y\beta + z = 0,$$

for $x, y, z \in \mathbb{R}$, where $x = 0$ if and only if both classes defining the wall have the same classical slope. In particular, they are either semicircles with center on the β -axis or vertical rays.

- (2) *Numerical walls only intersect if they are identical.*
(3) *If a numerical wall has a single point at which it is an actual wall, then all of it is an actual wall.*
(4) *The equation $Q_{\alpha,\beta}(v) = 0$ is a numerical wall for v .*

Theorem 4.2. *There exists $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that the line bundle $\mathcal{O}_X(L)$ is $\nu_{\alpha,\beta}$ -semistable, but $Q_{\alpha,\beta}(\mathcal{O}_X(L)) < 0$.*

Proof. Since $\mathcal{O}(L)$ is a line bundle it is a slope semistable sheaf. In particular, either $\mathcal{O}(L)$ or $\mathcal{O}(L)[1]$ is $\nu_{\alpha,\beta}$ -stable for all $\alpha \gg 0$.

A straightforward computation shows $H^3 \cdot \text{ch}_0(\mathcal{O}(L)) = 7$, $H^2 \cdot \text{ch}_1(\mathcal{O}(L)) = 4$, $H \cdot \text{ch}_2(\mathcal{O}(L)) = 1$ and $\text{ch}_3(\mathcal{O}(L)) = 1/6$. In particular, this means $H^2 \cdot \text{ch}_1^{1/2}(\mathcal{O}(L)) = 1/2$. If $F \hookrightarrow \mathcal{O}(L)$ destabilizes along the line $\beta = 1/2$, we must have $\text{ch}_1^{1/2}(F) \in \{0, 1/2\}$. That means either F or the quotient has slope infinity independently of α , a contradiction.

A completely numerical computation shows that $Q_{\alpha,\beta}(\mathcal{O}(L)) \geq 0$ is equivalent to the inequality

$$\alpha^2 + \left(\beta - \frac{1}{4}\right)^2 \geq \frac{1}{16}.$$

We are done if we can prove that there is no wall with equality in this inequality for $\mathcal{O}(L)$. Assume there is a destabilizing sequence $0 \rightarrow F \rightarrow \mathcal{O}(L) \rightarrow G \rightarrow 0$ giving exactly this wall. Taking the long exact sequence in cohomology, we get $H^3 \cdot \text{ch}_0(F) \geq 7$. By definition of $\text{Coh}^\beta(X)$ we have the inequalities $H^2 \cdot \text{ch}_1^\beta(\mathcal{O}(L)) \geq H^2 \cdot \text{ch}_1^\beta(F) \geq 0$ for all $\beta \in [0, 1/2]$. This can be rewritten as

$$4 + \beta(H^3 \cdot \text{ch}_0(F) - 7) \geq H^2 \cdot \text{ch}_1(F) \geq \beta H^3 \cdot \text{ch}_0(F).$$

Notice that the middle term is independent of β and we can vary β independently on the left and right. Therefore, we get $4 \geq H^2 \cdot \text{ch}_1(F) \geq H^3 \cdot \text{ch}_0(F)/2$. This means $H^2 \cdot \text{ch}_1(F) = 4$ and $H^3 \cdot \text{ch}_0(F) = 7$. This does not give the correct wall. \square

5. PROJECTIVE SPACE

We finish by giving a proof of the conjecture in the case of projective space \mathbb{P}^3 . By abuse of notation we will write $\text{ch}_i^\beta(E)$ instead of $H^{3-i} \text{ch}_i^\beta(E)$ for any $E \in D^b(\mathbb{P}^3)$. In this special case the Hirzebruch-Riemann-Roch Theorem says the following.

Theorem 5.1. *Let $E \in D^b(\mathbb{P}^3)$. Then*

$$\chi(\mathbb{P}^3, E) = \text{ch}_3(E) + 2 \text{ch}_2(E) + \frac{11}{6} \text{ch}_1(E) + \text{ch}_0(E)$$

holds.

Lemma 5.2 ([BMS14]). *It is enough to prove the inequality for objects $E \in \text{Coh}^{\beta_0}(X)$ such that $\text{ch}_2^{\beta_0}(E) = 0$ and E is stable in a neighborhood of $(\alpha, \beta) = (0, \beta_0)$. Moreover, in that case the inequality reduces to $\text{ch}_3^{\beta_0}(E) \leq 0$.*

The following proof idea is due to [Li15].

Proof of Conjecture for \mathbb{P}^3 . By the structure theorem for walls in tilt stability it is not hard to prove that line bundles are stable everywhere in tilt stability in the case $X = \mathbb{P}^3$. Let $E \in \text{Coh}^{\beta_0}(X)$ be an object such that $\text{ch}_2^{\beta_0}(E) = 0$ and E is stable in a neighborhood of $(\alpha, \beta) = (0, \beta_0)$. By tensoring with line bundles, we can assume $\beta_0 \in [-1, 0)$.

By assumption we have $\nu_{0, \beta_0}(E) = 0 < \nu_{0, \beta_0}(\mathcal{O}_{\mathbb{P}^3})$ which implies $\text{Hom}(\mathcal{O}_{\mathbb{P}^3}, E) = 0$. Moreover, the same argument together with Serre duality shows $\text{Ext}^2(\mathcal{O}_{\mathbb{P}^3}, E) = \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3}(-4)) = \text{Hom}(E, \mathcal{O}_{\mathbb{P}^3}(-4)[1]) = 0$. Therefore,

$$\begin{aligned} 0 \geq \chi(\mathcal{O}, E) &= \text{ch}_3(E) + 2 \text{ch}_2(E) + \frac{11}{6} \text{ch}_1(E) + \text{ch}_0(E) \\ &= \text{ch}_3^{\beta_0}(E) + (\beta_0 + 2) \text{ch}_2^{\beta_0}(E) + \frac{1}{6}(3\beta_0^2 + 12\beta_0 + 11) \text{ch}_1^{\beta_0}(E) \\ &\quad + \frac{1}{6}(\beta_0^3 + 6\beta_0^2 + 11\beta_0 + 6) \text{ch}_0^{\beta_0}(E) \\ &= \text{ch}_3^{\beta_0}(E) + \frac{1}{6}(3\beta_0^2 + 12\beta_0 + 11) \text{ch}_1^{\beta_0}(E) + \frac{1}{6}(\beta_0^3 + 6\beta_0^2 + 11\beta_0 + 6) \text{ch}_0^{\beta_0}(E). \end{aligned}$$

holds. By construction of $\text{Coh}^{\beta_0}(X)$, we have $\text{ch}_1^{\beta_0}(E) \geq 0$. Due to $\beta_0 \in [-1, 0)$ we also have $3\beta_0^2 + 12\beta_0 + 11 \geq 0$ and $\beta_0^3 + 6\beta_0^2 + 11\beta_0 + 6 \geq 0$. If $\text{ch}_0^{\beta_0}(E) \geq 0$ or $\beta_0 = -1$, this finishes the proof. If $\text{ch}_0^{\beta_0}(E) < 0$ and $\beta_0 \neq -1$, the same type of argument works with $\chi(\mathcal{O}(3), E) \leq 0$. \square

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